Lecture 4:

continuous charge distributions

shell theorem

Electric fields

continuous charge distributions

$$\sum_{i=1}^{N} q_i \rightarrow \int \lambda(x)\, ds(x)$$

$\lambda(x)$ charge / length at x

$$\rightarrow \int \sigma(x)\, dA(x)$$

$\sigma(x)$ charge / area at x

$$\rightarrow \int \rho(x)\, dV(x)$$

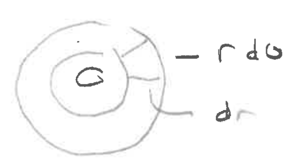
$\rho(x)$ charge / volume at x

volume integrals

$$\int dV = \int_0^L dx \int_0^w dy \int_0^h dz = Lwh =$$

volume of cube

$$\int dV = \int_0^h dz \int_0^R dr \int_0^{2\pi} r\, d\theta$$



$$\int dV = \int_0^R dr \int_0^{\pi} r\, d\theta \int_0^{2\pi} r\sin\theta\, d\varphi$$

sphere

If we do these integrals

$$V = \int dV = \int_0^R dr \int_0^{\pi} r\, d\theta \int_0^{2\pi} r\sin\theta\, d\phi$$

$$= \int_0^R dr \int_0^{\pi} r^2 \sin\theta \; 2\pi \; d\theta$$

$$= \int_0^R dr\, r^2 \cdot 2\pi \cdot \left(-\cos\pi - (-\cos 0)\right)$$

$$= \int_0^R 4\pi r^2\, dr = 4\pi\left(\frac{R^3}{3} - \frac{0}{3}\right) = \frac{4}{3}\pi R^3$$

which is the volume of a sphere of radius R.

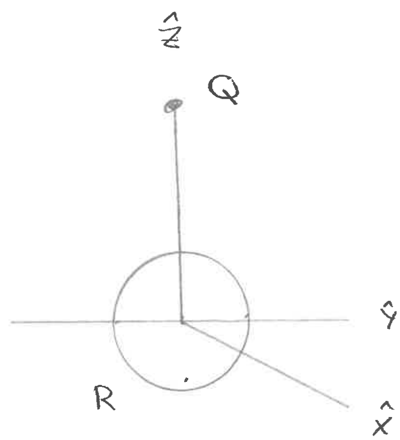
Next we compute the force on a particle of charge Q a distance r_Q from the center of a spherically symmetric charge distribution $\rho(\bar{r}) = \rho(r)$

The first step is to choose a useful coordinate system

(1) origin at center of sphere

(2) z axis along the line between the center of the sphere and the charge Q

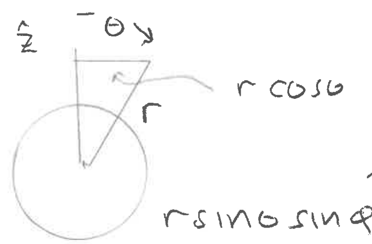
(3) use spherical polar coordinates

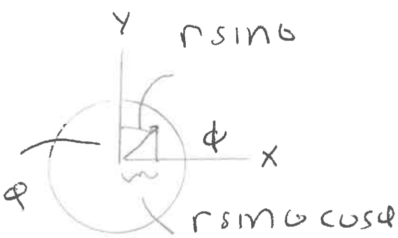


In this coordinate system

$$\bar r = (0, 0, r_Q)$$

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$





using coulomb's law and the superposition principle gives

$r\sin\theta\cos\phi$, $r\sin\theta\sin\phi$, $r\cos\theta$

$$\bar F_Q(r_Q) = kQ \underbrace{\int_0^R r^2\rho(r)\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi}_{\int \rho\, dV}\ \frac{(0,0,r_Q)-(x,y,z)}{\left|\sqrt{r^2-2rr_Q\cos\theta+r_Q^2}\right|^3}$$

where we used

$$|\bar r-(x,y,z)|^2 = (r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + (r_Q-r\cos\theta)^2)$$

$$= r^2\sin^2\theta\underbrace{(\cos^2\phi+\sin^2\phi)}_{1} + r^2\cos^2\theta - 2rr_Q\cos\theta + r_Q^2$$

$$= r^2\underbrace{(\sin^2\theta+\cos^2\theta)}_{1} - 2rr_Q\cos\theta + r_Q^2$$

$$= r^2 - 2rr_Q\cos\theta + r_Q^2$$

In computing the integrals there is a separate integral for each component of $\bar{F}$

$$F_x = kQ\int_0^R r^2 \rho(r) \int_0^\pi \sin\theta d\theta \int_0^{2\pi} \frac{-r\sin\theta\cos\phi}{(r^2 - 2rr_0\cos\theta + r_0^2)^{3/2}} d\phi$$

the only ϕ dependence in this integral is from $\cos\phi$

$$\int_0^{2\pi} \cos\phi \, d\phi = \sin(2\pi) - \sin(0) = 0 - 0 = 0$$

$$F_y = kQ\int_0^R r^2 \rho(r) \int_0^\pi \sin\theta d\theta \int_0^{2\pi} \frac{-r\sin\theta\sin\phi}{(r^2 - 2rr_0\cos\theta + r_0^2)^{3/2}} d\phi$$

$$\int_0^{2\pi} \sin\phi \, d\phi = -\cos(2\pi) - (-\cos(0)) = -(1) - (-1) = 0$$

so the x and y components of the force vanish in this coordinate system.

$$F_z = kQ\int_0^R r^2 dr \, \rho(r) \int_0^\pi \sin\theta d\theta \cdot \int_0^{2\pi} \frac{(r_0 - r\cos\theta)}{(r^2 - 2rr_0\cos\theta + r_0^2)^{3/2}} d\phi$$

in this case the integrand is independent of ϕ so the ϕ integral gives 2π.

To compute the θ integral we use

$$\frac{\partial}{\partial r_Q} \frac{1}{\sqrt{r^2 - 2rr_Q\cos\theta + r_Q^2}} =$$

$$\left(-\frac{1}{2}\right) \frac{2r_Q - 2r\cos\theta}{\left(\sqrt{r^2 - 2rr_Q\cos\theta + r_Q^2}\right)^3} =$$

$$\frac{r\cos\theta - r_Q}{\left(\sqrt{r^2 - 2rr_Q\cos\theta + r_Q^2}\right.}$$

This is - the integrand in the θ integral.

$$F_z = 2\pi k Q \int_0^R r^2 dr\, \rho(r) \left(-\frac{d}{dr_Q}\right) \int_0^\pi \frac{\sin\theta\, d\theta}{\sqrt{r^2 - 2rr_Q\cos\theta + r_Q^2}}$$

To do the θ integral let

$$u(\theta) = r^2 - 2rr_Q\cos\theta + r_Q^2$$

$$du = \frac{du}{d\theta} d\theta = -(2rr_Q)(-\sin\theta)d\theta = 2rr_Q\sin\theta\, d\theta$$

$$\sin\theta\, d\theta = \frac{1}{2rr_Q} du$$

$$u(\pi) = r^2 - 2rr_Q\cos(\pi) + r_Q^2 = r^2 + 2rr_Q + r_Q^2 = (r + r_Q)^2$$

$$u(0) = r^2 - 2rr_Q\cos(0) + r_Q^2 = r^2 - 2rr_Q + r_Q^2 = (r - r_Q)^2$$

with these substitutions

$$F_z(r_Q) = 2\pi k Q \int_0^R \rho(r) r^2 dr \left(-\frac{d}{dr_Q}\right) \frac{1}{2rr_Q} \int_{|r-r_Q|^2}^{|r+r_Q|^2} u^{-1/2} du$$

$$= 2\pi k Q \int_0^R r^2 \rho(r) dr \left(-\frac{d}{dr_Q}\right) \frac{1}{2rr_Q} 2(|r+r_Q| - |r-r_Q|)$$

there are 2 cases of interest

case 1 $r_Q > R$ - in that case $r_Q > r$

since $r: 0 \to R$ and $|r+r_Q| - |r-r_Q| = r_Q + r - r_Q + r = 2r$

then

$$-\frac{d}{dr_Q} \frac{1}{2rr_Q} \cdot 2(|r+r_Q| - |r-r_Q|) = -\frac{d}{dr_Q} \frac{2}{r_Q} = \frac{2}{r_Q^2}$$

then

$$F_z(r_Q) = \frac{2\pi k Q \cdot 2}{r_Q^2} \int_0^R r^2 \rho(r) \, dR$$

note that

$$Q_s = \int_0^R \rho(r) r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi = 4\pi \int_0^R r^2 \rho(r) \, dr$$

using this

$$F_z(r_Q) = \frac{k Q Q_s}{r_Q^2}$$

using the rest of the components

$$\vec{F}(r_Q) = \left(0, 0, \frac{k Q Q_s}{r_Q^2}\right)$$

This is exacly the Coulomb force on a particle of charge Q a distance r_Q from a point charge with the total charge on the sphere. The result is <u>independent</u> of $\rho(r)$. - all that matters is the total charge

Next consider the case where $r_Q < R$. In this case we write

$$\int_0^R dr = \int_0^{r_Q} dr + \int_{r_Q}^R dr$$

for the first integral $r < r_Q$ and the result is the same as above except

$$4\pi \int_0^R r^2 \rho(r)\, dr$$

is replaced by

$$4\pi \int_0^{r_Q} \rho(r)\, r^2\, dr$$

which is the total charge in the part of the spherical charge distribution with $r < r_Q$. $Q(r_Q)$

For the second integral $r > r_Q$

$$-\frac{d}{dr_Q}\frac{1}{2rr_Q}2\left(|r+r_Q|-|r-r_Q|\right) =$$

$$-\frac{d}{dr_Q}\frac{1}{2rr_Q}2\left(r+r_Q-(r-r_Q)\right) =$$

$$-\frac{d}{dr_Q}\frac{4r_Q}{2rr_Q} = -\frac{d}{dr_Q}\left(\frac{2}{r}\right) = 0$$

since $\left(\frac{2}{r}\right)$ is independent of r_Q.

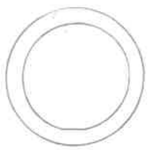
The means that the second integral gives "0"

It follow that for $r_Q < R$

$$F(r_Q) = \left(0, 0, \frac{kQQ_s(r_Q)}{r_Q^2}\right)$$

Total charge in the part of the sphere with $r < r_Q$.

Implications

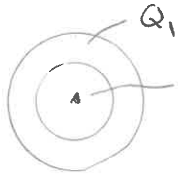


← charged shell with charge Q_s

No force on particle inside of shell,

Force on particle outside of shell is

$$\bar{F} = \frac{kQQ_s}{r^2}\hat{r}$$



Total charge on sphere is $Q_+ + Q_- = 0$ – no force outside; but there is a force inside where the inner charge is $-Q_1$

This is the content of the shell theorem – the force on a particle of charge Q due to a spherically symmetric charge distribution is the same as the force due to a point charge of total charge = charge in sphere for $r < r_0$.

This is a special case of something we will show later called Gauss law

Electric Fields

Consider the force on a particle of charge Q to q_1 at $r_1 \cdots q_N$ at r_N

$$\bar{F}(\bar{r}) = \sum_{n=1}^{N} kQq_n \frac{\bar{r}-\bar{r}_n}{|\bar{r}-\bar{r}_n|^3} =$$

$$= Q\left(\sum_{n=1}^{N} kq_n \frac{\bar{r}-\bar{r}_n}{|\bar{r}-\bar{r}_n|^3}\right)$$

$$= Q\,\bar{E}(\bar{r})$$

depends on the particle at $\bar{r}$

depends on all of the other particles

The vector field $\bar{E}(\bar{r})$ is called the electric field. It defines a vector at each point in space.

What we mean by a field is a property that depends on coordinates

(1) $T(\bar{r})$ = temperature at $\bar{r}$ is a <u>scalar</u> field

(2) $\bar{V}(\bar{r})$ = wind velocity at $\bar{r}$ is a <u>vector</u> field

(3) $\bar{E}(\bar{r})$ = is a vector field

the electric field exists at each point $\bar{r}$, even if there is no particle there.

experimentally if a charge Q is placed at $\bar{r}$ the other charges will experience a force due to that charge. To prevent this the value of the field at $\bar{r}$ is defined by

$$\boxed{\bar{E}(\bar{r}) \equiv \lim_{Q \to 0} \frac{\bar{F}_Q(\bar{r})}{Q}}$$

The field due to a point charge Q at the origin

$$\bar{F}_Q(\bar{r}) = kQQ_0 \frac{\bar{r}}{|r|^3}$$

$$\bar{E}(r) = \lim_{Q_0 \to 0} \frac{\bar{F}_Q}{Q_0} = \frac{kQQ_0}{Q_0} \frac{\bar{r}}{|r|^3} = kQ\frac{\bar{r}}{|r|^3}$$

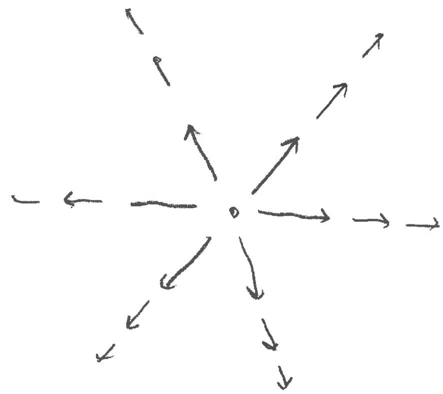
Just like force the field obeys the superposition principle, the field at $\bar{r}$ due to a collection of point charges is

$$\bar{E}(\bar{r}) = \sum \bar{E}_i(\bar{r}) = \sum_{n=1}^{N} kq_i \frac{\bar{r}-\bar{r}_i}{|r-r_i|^3}$$

We can also generalize to continuous charge distributions

$$\boxed{\bar{E}(\bar{r}) = \int k\rho(\bar{r}') \frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|^3} dV'}$$

Fields can be visualized by drawing a vector at each point with length given by the magnitude of the field at that point



The arrows give the direction of the force on a positive charge placed at that point.

Field due to a point dipole (note error corrected in lecture 3)

$$\bar{E} = \frac{1}{Q}(Q(-\bar{\nabla}))\left(\frac{k\bar{p}\cdot\bar{r}}{r^3}\right)$$

forgot $\frac{1}{r}$ last time

$$= -\nabla\left(\frac{kPZ}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= -\left(-\frac{3}{2}\right)\left(\frac{2PkxZ}{r^{5/2}}, \frac{2PkyZ}{r^{5/2}}, \frac{2PkZ^2}{r^{5/2}}\right)$$

$$- \frac{(0, 0, kPr^2)}{r^5}$$

$$= \frac{3k\bar{p}\cdot\bar{r}}{r^5} - \frac{k\bar{p}r^2}{r^5} =$$

$$= \frac{3k\,\bar{p}\cdot\bar{r} - \bar{p}r^2}{r^5}$$