Lecture 5

Electric field at a point $\vec{r}$ due to a distribution of changes

$$\boxed{\bar{E}(\bar{r}) = k \sum_{n=1}^{N} q_n \frac{\bar{r}-\bar{r}_n}{|\bar{r}-\bar{r}_n|^3}}$$

$\bar{r}_n$ = position of change q_n

For continuous change distributions

$$\bar{E}(\bar{r}) = k\int \lambda(\bar{r}')\,ds' \frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|^3}$$

or

$$= k\int \sigma(\bar{r}')\,dA' \frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|^3}$$

$$= k\int \rho(\bar{r}')\,dV' \frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|^3}$$

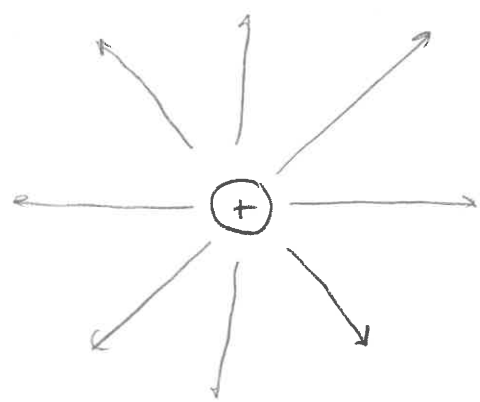
If a change q at position $\bar{r}$ is placed in an electric field $\bar{E}(\bar{r})$ it will experience a force

$$\boxed{\bar{F} = q\,\bar{E}(\bar{r})}$$

The field can be expressed graphically by drawing an arrow at $\bar{r}$ pointing in the same direction as $\bar{E}(\bar{r})$

with length proportional to the magnitude of $\bar{E}$, the units of the field are Newton's / Coulomb.

Electric field lines are lines coming in or out of charges with the tangent vector to the curve at $(\bar{r})$ pointing in the same direction as the field at $\bar{r}$



$$\bar{E}(\bar{r}) = kq \frac{\hat{r}}{r^2}$$

for a charge q at the origin

Field due to a dipole with dipole moment $\bar{p}$

$$\bar{F} = kq(-\bar{\nabla})\left(\frac{\bar{p}\cdot\bar{r}}{r^3}\right)$$

$$\bar{E}(\bar{r}) = \lim_{q\to 0} \frac{\bar{F}(\bar{r})}{q} = -k\bar{\nabla}\left(\frac{\bar{p}\cdot\bar{r}}{r^3}\right)$$

where

$$\bar{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

the chain rule gives

$$\bar{E}(\bar{r}) = -k\,\bar{\nabla}(\bar{p}\cdot\bar{r})\frac{1}{r^3} - k(\bar{p}\cdot\bar{r})\bar{\nabla}\frac{1}{r^3}$$

$$\bar{\nabla}(\bar{p}\cdot\bar{r}) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)(P_x X + P_y Y + P_z Z)$$

$$= \hat{x}P_x + \hat{y}P_y + \hat{z}P_z$$

$$= \vec{P}$$

$$\nabla\frac{1}{r^3} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{-3/2}$$

$$= -\frac{3}{2}(x^2+y^2+z^2)^{-5/2}(2x\hat{x} + 2\hat{y}y + 2\hat{z}z)$$

$$= -3\frac{\vec{r}}{r^5}$$

combining these give the formula for the field due a dipole with dipole moment $\vec{p}$

$$E(\bar{r}) = -k\frac{\bar{p}}{r^3} + -k(-3\,\bar{p}\cdot\bar{r})\frac{\bar{r}}{r^5}$$

$$= -k\frac{\bar{p}r^2}{r^5} + k\,\frac{3\bar{r}(\bar{p}\cdot\bar{r})}{r^5}$$

$$\boxed{\bar{E}(r) = k\,\frac{3\bar{r}(\bar{r}\cdot\bar{p}) - r^2\bar{p}}{r^5}}$$

Electric field due to a point dipole with dipole moment $\bar{p}$.

If $\bar{p}$ is in the $\hat{z}$ direction $\bar{p} = p\hat{z}$

$$\boxed{\bar{E} = kp\left(\frac{3\bar{r}\,z}{r^5} - \frac{r^2\hat{z}}{r^5}\right)}$$

A characteristic property of a dipole field is that it falls off like $1/r^3$

Electric field due to an infinite line charge

Choose coordinates so the line charge, with charge/length λ is along the z axis. We assume λ is constant

$$\bar{E}(r) = k\lambda \int_{-\infty}^{\infty} dz' \frac{\bar{r}-(0\,0\,z')}{|r-(0\,0\,z')|^3}$$

note

$$\bar{r}-(0,0,z) = (x, y, z-z')$$

$$|\bar{r}-(0\,0\,z)| = \sqrt{x^2+y^2+(z-z')^2}$$

$$\bar{E}(\bar{r}) = k\lambda \int_{-\infty}^{\infty} \frac{(x, y, (z-z'))}{\left(\sqrt{x^2+y^2+(z-z')^2}\right)^3}$$

let $z'' = z-z'$ $\quad dz'' = -dz'$ $\quad z'': \infty \to -\infty$

$$= k\lambda \int_{\infty}^{-\infty} \frac{(x, y, z'')}{\left(\sqrt{x^2+y^2+z''^2}\right)^3}(-dz'')$$

$$= k\lambda \left(-\int_{-\infty}^{+\infty}\right) \frac{(x, y\, z'')}{\left(\sqrt{x^2+y^2+z''^2}\right)^3}(-dz'')$$

Note

$$\int_{-\infty}^{\infty} \frac{z'' dz''}{(x^2+y^2+z''^2)^{3/2}} =$$

$$\int_{-\infty}^{0} \frac{dz'' z''}{(x^2+y^2+z''^2)^{3/2}} + \int_{0}^{\infty} \frac{dz'' z''}{(x^2+y^2+z''^2)}$$

let $z''' = -z''$ in the first integral

$dz'' = -dz'''$ $\quad z''dz'' = (-z''')(-dz''') = z'''dz'''$

$$\left(\int_{\infty}^{0} + \int_{0}^{\infty}\right) \frac{dz'' z''}{(x^2+y^2+z''^2)^{3/2}} = 0$$

this is because we are integrating an odd function

For the x and y components

$$E_x(\bar{r}) = k\lambda x \int_{-\infty}^{\infty} \frac{dz'}{(x^2+y^2+(z-z')^2)^{3/2}} \qquad \text{let } z'' = (z'-z)$$

$$= k\lambda x \int_{-\infty}^{\infty} \frac{dz''}{(\rho^2+z''^2)^{3/2}} \qquad \rho^2 \equiv x^2+y^2$$

we do this integral using hyperbolic substitution

$$\cosh \eta = \tfrac{1}{2}(e^{\eta}+e^{-\eta})$$

$$\sinh \eta = \tfrac{1}{2}(e^{\eta}-e^{-\eta})$$

$$\frac{d\cosh \eta}{d\eta} = \tfrac{1}{2}(e^{\eta}-e^{-\eta}) = \sinh \eta$$

$$\frac{d\sinh \eta}{d\eta} = \tfrac{1}{2}(e^{\eta}-(-)e^{-\eta}) = \cosh \eta$$

$$\cosh^2\eta - \sinh^2\eta =$$

$$\tfrac{1}{4}(e^{2\eta}+2+e^{-2\eta}) - \tfrac{1}{4}(e^{2\eta}-2+e^{-2\eta}) = 1$$

$$\tanh\eta = \frac{\sinh\eta}{\cosh\eta}$$

$$\frac{d\tanh\eta}{d\eta} = \frac{\cosh\eta}{\cosh\eta} - \frac{\sinh\eta\,\sinh\eta}{\cosh^2\eta}$$

$$= 1 - \frac{\sinh^2\eta}{\cosh^2\eta} = \frac{\cosh^2\eta - \sinh^2\eta}{\cosh^2\eta}$$

$$= \frac{1}{\cosh^2\eta}$$

In the integral let $z = \rho\sinh\eta$

as $z: -\infty \to \infty$ $\quad \eta: -\infty \to \infty$

$$z = \rho\sinh\eta$$
$$dz = \rho\cosh\eta\, d\eta$$

$$\frac{1}{(\rho^2+z^2)^{3/2}} = \frac{1}{(\rho^2(1+\sinh^2\eta))^{3/2}} = \frac{1}{(\rho^2\cosh^2\eta)^{3/2}} =$$

$$\frac{1}{\rho^3\cosh^3\eta}$$

$$E_x(\bar{r}) = k\lambda x\int_{-\infty}^{\infty}\frac{\rho\cosh\eta\, d\eta}{\rho^3\cosh^3\eta} =$$

$$= k\lambda x\frac{1}{\rho^2}\int_{-\infty}^{\infty}\frac{d\eta}{\cosh^2\eta}$$

$$= k\lambda\frac{x}{\rho^2}\int_{-\infty}^{\infty}\frac{d}{d\eta}(\tanh\eta)$$

Note $\int_{-\infty}^{\infty} \frac{z'' dz''}{(x^2+y^2+z''^2)^{3/2}} =$

$$\int_{-\infty}^{0} \frac{dz'' z''}{(x^2+y^2+z''^2)^{3/2}} + \int_{0}^{\infty} \frac{dz'' z''}{(x^2+y^2+z''^2)^{3/2}} =$$

$z'' = -z'''$ $\quad dz'' = -dz'''$

$$-\int_{0}^{\infty} \frac{dz''' z'''}{(x^2+y^2+z'''^2)^{3/2}} + \int_{0}^{\infty} \frac{dz'' z''}{(x^2+y^2+z''^2)^{3/2}} = 0$$

This means that $E(\bar{r})$ has <u>no</u> z component

For the x or y component

$$E_x(\bar{r}) = k\lambda x \int_{-\infty}^{\infty} \frac{dz'}{(x^2+y^2+(z-z')^2)^{3/2}} \qquad z'' = z - z'$$

$$= k\lambda x \int_{-\infty}^{\infty} \frac{dz''}{(x^2+y^2+z''^2)^{3/2}} \qquad \rho^2 = x^2+y^2$$

$$= -k\lambda x \frac{1}{\rho}\frac{\partial}{\partial \rho} \int_{-\infty}^{\infty} \frac{dz}{(\rho^2+z'^2)^{1/2}} \qquad z = \rho \sinh \eta$$

$\rho^2 + z^2 =$

$\rho^2 + \rho^2 \sinh^2 \eta =$

$\rho^2(1+\sinh^2\eta)$

$\rho^2 \cosh^2 \eta$

$dz = \rho \cosh \eta \, d\eta$

$$\frac{1}{(\rho^2+z^2)^{1/2}} = \frac{1}{\rho \cosh \eta}$$

$$= -k\lambda x \frac{1}{\rho}\frac{\partial}{\partial \rho} \int_{-\infty}^{\infty}$$

$$k\lambda \frac{x}{\rho^2}\left(\frac{e^{\eta}-e^{-\eta}}{e^{\eta}+e^{-\eta}}\right)\Bigg|_{-\infty}^{\infty} = k\lambda\frac{x}{\rho}\left(1-(-1)\right) = 2k\lambda\frac{x}{\rho}$$

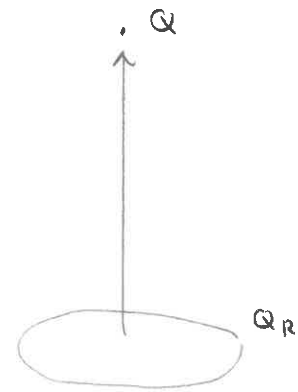
The calculation for $E_y(\bar{r})$ is the same

$$\boxed{\bar{E}(\bar{r}) = 2k\lambda\left(\frac{x,y,0}{\rho^2}\right) = 2k\lambda\frac{\hat{\rho}}{\rho}}$$

where $\hat{\rho} = \frac{(x,y)}{\rho} = \frac{(x,y)}{\sqrt{x^2+y^2}}$ is the unit vector in the radial direction

Field due to a ring and disk

Friday



$$\bar{F}(0,0,z) = Q\cdot Q_R\, k \frac{z}{(R^2+z^2)^{3/2}}\hat{z}$$

$$\bar{E}(0,0,z) = \frac{\bar{F}(0,0,z)}{Q} = kQ_R\frac{z}{(R^2+z^2)^{3/2}}\hat{z}$$

$$= k\frac{Q_R}{2\pi R}\cdot\frac{2\pi R\, z}{(R^2+z^2)^{3/2}} = 2\pi R\lambda k\frac{z}{(R^2+z^2)^{3/2}}$$

we can replace the ring by a disk of radius R by summing up the contributions from

For each ring

$$dE = k\,dq \frac{z}{(r^2+z^2)^{3/2}} \qquad r = \text{radius of ring}$$

$$= k \frac{dq}{dr} \frac{z}{(r^2+z^2)^{3/2}}\,dr$$

$$dq = \sigma\, dA = 2\pi r \sigma\, dr$$

$$\bar{E}(z) = \int_0^R k\, 2\pi r \sigma \frac{z}{(z^2+r^2)^{3/2}}\,dr$$

$$= 2\pi k z \sigma \int_0^R \frac{r}{(z^2+r^2)^{3/2}}\,dr$$

Let $u = z^2 + r^2 \qquad du = 2r\,dr$

$$= 2\pi k z \sigma \int_{z^2}^{z^2+R^2} \tfrac{1}{2}\, u^{-3/2}\,du$$

$$= 2\pi k z \sigma \cdot \tfrac{1}{2} \cdot (-2)\left(u^{-1/2} \right)\Big|_{z^2}^{z^2+R^2}$$

$$= 2\pi k z \sigma \cdot (-)\left(\frac{1}{\sqrt{z^2+R^2}} - \frac{1}{z} \right)$$

$$\boxed{\underset{(0,0,z)}{\bar{E}(z)} = 2\pi k \sigma \left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)\hat{z}}$$

Field due to disk along z axis

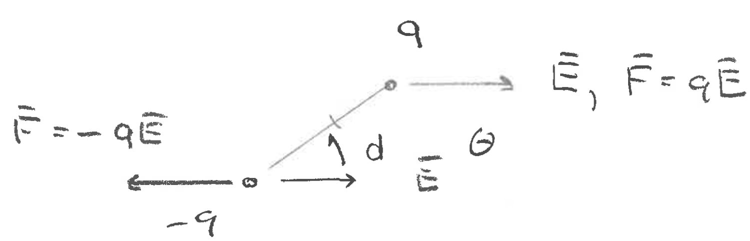
From this expression we can calculate the field due to an ∞ plane with charge / area $= \sigma$ by letting $R \to \infty$

$$\bar{E} = \lim_{R \to \infty} 2\pi k \sigma \left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)\hat{z} =$$

$$\boxed{\bar{E} = 2\pi k \sigma \hat{z}}$$

In this case the electric field is constant, independent of distance from the plane

Force on an electric dipole in a uniform electric field



In this case we that the net force on the dipole is 0 because it is electrically neutral; but it will experience a torque about its center.

$$T = qE\frac{d}{2}\sin\theta - qE\left(-\frac{d}{2}\right)\sin\theta$$

$$= qEd\sin\theta = pE\sin\theta$$

The direction of the torque in into the plane of the paper

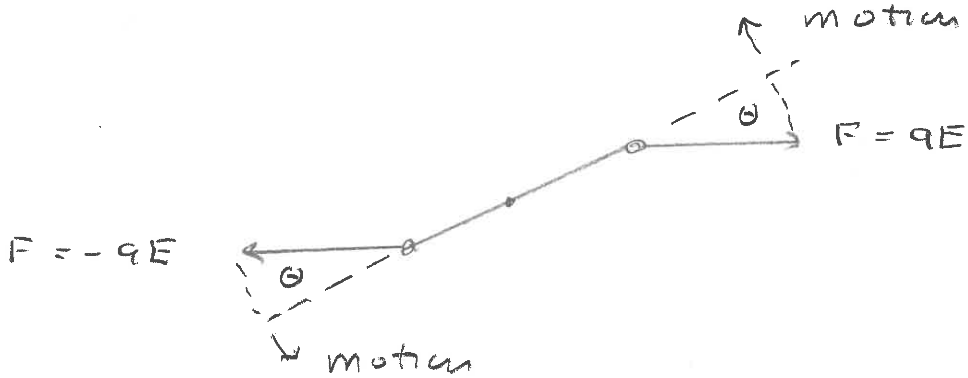
$$\bar{p} \times \bar{E}$$

since $|\hat{p} \times \bar{E}| = pE\sin\theta$

$$\boxed{\vec{T} = \bar{p} \times \bar{E}}$$

This torque will vanish when the dipole points parallel to the field.

Work done by <u>field</u> on a dipole



$$dW_+ = q\bar{E}\cdot\left(-\frac{d}{2}\sin\theta\right)d\theta$$

$$dW_- = (-q\bar{E})\left(\frac{d}{2}\sin\theta\right)d\theta$$

$$dW_T = dW_+ + dW_- = -q\bar{E}\,d\sin\theta\,d\theta$$

$$W = qEd\cos\theta = EP\cos\theta = \bar{E}\cdot\bar{P}$$

$W = -\mathcal{U}$ = potential

$$\mathcal{U} = -\bar{E}\cdot\bar{P}$$