Lecture 6

Infinite line charge

λ = charge / length

$$\vec{E}(\vec{r}) = k\lambda \int_{-\infty}^{\infty} dz' \frac{(x\ y\ z) - (0,0\ z')}{(x^2 + y^2 + (z-z')^2)^{3/2}}$$

(this uses Coulombs law and the superposition principle.)

Let $z'' = z - z'$ $\quad dz'' = -\,dz'$

$$= k\lambda \int_{+\infty}^{-\infty} (-dz'') \frac{(x, y, z'')}{(x^2+y^2+z''^2)^{3/2}} =$$

$$k\lambda \int_{-\infty}^{+\infty} dz'' \frac{(x, y, z'')}{(x^2+y^2+z''^2)^{3/2}}$$

The integral for the z component vanishes because

$$f(z'') = \frac{z''}{(x^2+y^2+z''^2)^{3/2}}$$

is odd $-$ $f(-z'') = -f(z'')$ $\qquad$ let $z''' = -z''$

$$\int_{-\infty}^{\infty} f(z'')\,dz'' = \int_{-\infty}^{0} f(z'')\,dz'' + \int_{0}^{\infty} f(z'')\,dz'' =$$

$$-\int_{\infty}^{0} dz''' f(-z''') + \int_{0}^{\infty} f(z'')\,dz'' = -\int_{0}^{\infty} dz''' f(z''') + \int_{0}^{\infty} f(z'')\,dz'' = 0$$

For the x and y components let

$x^2+y^2=r^2$ — $r = \perp$ distance from the z axis.

$$E_x = k\lambda x \int_{-\infty}^{\infty} \frac{dz''}{(r^2+z''^2)^{3/2}}$$

To do this integral recall

$$\cosh x = (e^x + e^{-x})/2$$

$$\sinh x = (e^x - e^{-x})/2$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\frac{d\cosh x}{dx} = \sinh x$$

$$\frac{d\sinh x}{dx} = \cosh x$$

$$\frac{d\tanh x}{dx} = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = 1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

Let $z'' = r\sinh u$ $\quad dz'' = r\cosh u \, du$

$u: -\infty \rightarrow \infty$

$r^2 + z''^2 = r^2(1+\sinh^2 u) = r^2\cosh^2 u$

$$\therefore \quad E_x = k\lambda x \int_{-\infty}^{\infty} \frac{r\cosh u}{(r\cosh u)^3}\,du =$$

$$= k\lambda x \int_{-\infty}^{\infty} \frac{du}{r^2\cosh^2 u} =$$

$$= k\lambda \frac{x}{r^2} \int_{-\infty}^{\infty} \frac{d}{du} \tanh u \cdot du$$

$$= k\lambda \frac{x}{r^2} \left(\tanh(\infty) - \tanh(-\infty) \right)$$

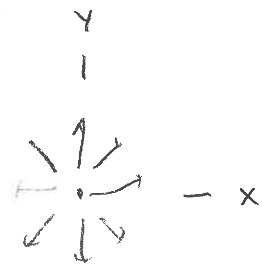
$$= k\lambda \frac{x}{r^2} \left(\underbrace{\frac{e^{\infty}-e^{-\infty}}{e^{\infty}+e^{-\infty}}}_{1} - \underbrace{\frac{e^{-\infty}-e^{\infty}}{e^{-\infty}+e^{\infty}}}_{-1} \right)$$

$$= 2k\lambda \frac{x}{x^2+y^2}$$

similarly

$$E_y = 2k\lambda \frac{y}{x^2+y^2}$$

$$\boxed{\vec{E} = 2k\lambda \frac{(x,y)}{(x^2+y^2)}}$$

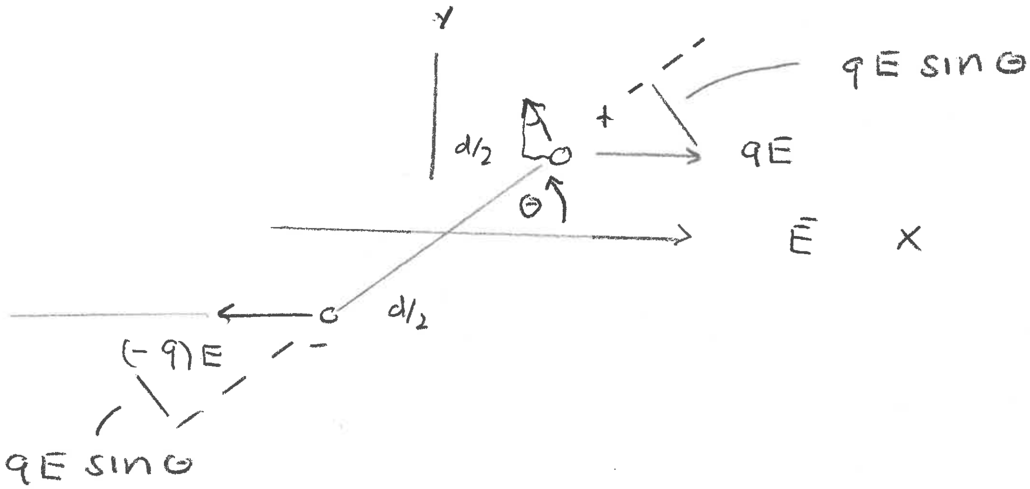


the field is directed radially outward and falls off like $1/r$ for large distances

Torque on an electric dipole in a constant electric field

recall

$$\bar{\tau} = \text{Torque} = \bar{r} \times \bar{F}$$



Torque – into plane of paper $(-\hat{z})$

$$\bar{T} = \frac{d}{2}(\hat{x}\cos\theta + \hat{y}\sin\theta) \times (qE\hat{x}) +$$

$$\frac{d}{2}(-\hat{x}\cos\theta - \hat{y}\sin\theta) \times (-qE\hat{x}) =$$

$$= 2\frac{d}{2} qE\sin\theta\,(\hat{y}\times\hat{x})$$

$$= (qd)E\sin\theta\,(-\hat{z})$$

$$= pE\sin\theta\,(-\hat{z})$$

Work done by field

(1) The net force on the dipole is 0. because it is electrically neutral

(2) Work is done rotating dipole

Work done by field

q^+ $\quad \bar{F}_+ = qE\hat{x}$

$\bar{r}_+ = (\hat{x}\frac{d}{2}\cos\theta + \hat{y}\frac{d}{2}\sin\theta)$

$F_- = (-q)E\hat{x}$

$\bar{r}_- = (-\hat{x}\frac{d}{2}\cos\theta - \hat{y}\frac{d}{2}\sin\theta)$

$d\bar{r}_+ = (\hat{x}\frac{d}{2}(-\sin\theta) + \hat{y}\frac{d}{2}\cos\theta)(-d\theta)$

$d\bar{r}_- = (-x\frac{d}{2}(-\sin\theta) - \hat{y}\frac{d}{2}\cos\theta)(-d\theta)$

$dW = \bar{F}_+ \cdot d\bar{r}_+ + \bar{F}_- \cdot d\bar{r}_- = 2\bar{F}_+ \cdot d\bar{r}_+$

$= 2qE\frac{d}{2}\sin\theta\, d\theta$

$= EP\sin\theta\, d\theta$

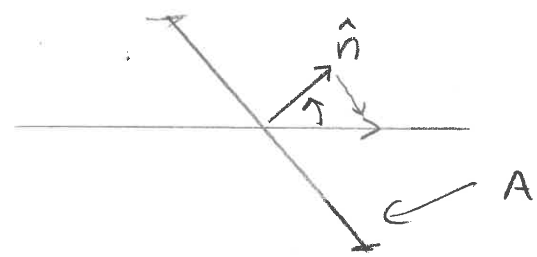
$W = EP(-\cos\theta)$

$W = -EP\cos\theta = -\bar{E}\cdot\bar{P} = U$

This is the potential energy of the dipole – it does negative work to get to the lowest potential energy – when $\theta = 0$ and the dipole is parallel to the field.

Electric Flux

consider water flowing with a velocity $\bar{V} = v\hat{x}$



assume the water is going through the slanted hole above, where $\hat{n}$ is the normal vector to the area.

The water velocity vector can be decomposed into a part parallel to the area and a part perpendicular to the area.

Water flowing parallel to the surface will not go through while water moving $\perp$ to the area will move through.

The volume of water that passes through the area is

$$\frac{dV_{ole}}{dt} = A \cdot \frac{dx_{\perp}}{dt} = A\hat{n}\cdot\vec{V}$$

$A\hat{n}\cdot\bar{V}$ is the <u>flux</u> of the fluid through the area A.

In general

$$\Phi = A\hat{n}\cdot\bar{V}$$

$$d\Phi = dA(x,y,z)\cdot\hat{n}\cdot\bar{V}(x,y,z)$$

so to comput the flux we have to integrate over the area.

There is nothing special about the velocity vector field – we can define flux for any vector field

consider a small area dA
with normal $\hat{n}$

The electric flux through dA is

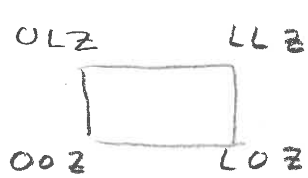
$$\boxed{d\Phi = \bar{E} \cdot \hat{n} \, dA}$$

<u>example 1</u>

Let $\bar{E} = E\hat{z}$ be a constant
electric field in the z direction.
(This is the field due to an
infinite plane with charge
density σ

$$\boxed{\bar{E} = 2\pi k \sigma \, \hat{z}}$$

consider a square with edges



The area of this square is L^2
the normal is in the $\pm\hat{z}$ direction

In this case the flux is

$$\Phi = \int_0^L dx \int_0^L dy \, (\pm \hat{z}) \cdot \bar{E}(x\,y\,z)$$

$$= \pm L^2 E$$

In this case there is some ambiguity with how we want to define the sign of the normal vector. For closed surfaces we usually choose the normal to be directed outward

Flux in a spherical surface due to a point charge Q at the origin

$$\bar{E} = kq \frac{\hat{r}}{r^2}$$

The surface area of a sphere of radius R is

$$dA = (R\,d\theta)(R\sin\theta\,d\varphi)$$

$$\int dA = \int_0^{\pi} \sin\theta\,d\theta \int_0^{2\pi} d\varphi \; R^2$$

$$2\pi R^2 \left(-\cos(\pi) - (-\cos 0)\right) =$$

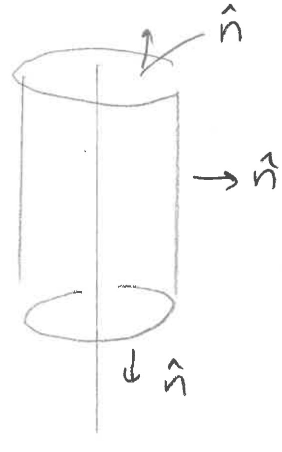
$= 4\pi R^2$

The normal (outward) to the surface of a sphere is the radial vector $\hat{r}$. The flux through the entire sphere is

$$\boxed{\begin{aligned}\Phi &= \bar{E}\cdot\hat{n}\,A = kq\frac{\hat{r}}{R^2}4\pi R^2\cdot\hat{r} \\ &= 4\pi k q = 4\pi\frac{q}{4\pi\epsilon_0} = \frac{q}{\epsilon_0}\end{aligned}}$$

here we see the flux depends on the charge

Next consider the flux through a cylinder of radius R and height h surrounding an infinite line charge



$\bar{E} = 2k\lambda\frac{\hat{r}}{r}$

There is no flux through the circular ends because $\hat{n} \perp \hat{r}$

for the outer surface

$$A = 2\pi R \cdot h$$

the normal is parallel to the field

$$\Phi = (2\pi R h)\left(2k\lambda \frac{1}{R}\right)$$

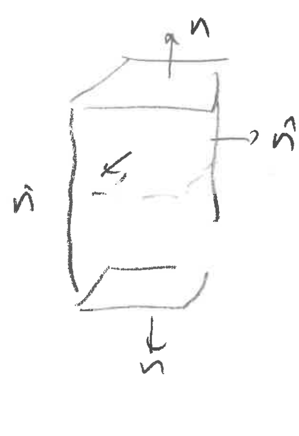
$$= 4\pi k(\lambda h) \qquad \lambda h = Q \text{ in cylinder}$$

$$= 4\pi k Q_c$$

$$\Phi = \frac{Q_c}{\epsilon_0}$$

The result is the total charge in the cylinder divided by ϵ_0.

consider the infinite plane again



$\uparrow \; 2\pi k\sigma \, \hat{n}$

infinite charged plane

$\downarrow \; 2\pi k\sigma \, \hat{n}$

here we consider the flux through a cube of side L^3 where the charged plane cuts through the middle

For the top and bottom the flux is

$$\Phi_T = L^2 \, 2\pi k \sigma$$

$$\Phi_B = L^2 \, 2\pi k \sigma$$

The sides do not contribute because the field is $\perp$ to the normals

so the total flux is

$$\Phi = \Phi_T + \Phi_B = 4\pi k \sigma L^2 = 4\pi k Q_\square$$

$$= \frac{Q_\square}{\epsilon_0}$$

where $Q_\square$ is the charge in the square.

We see in all three examples the result is the enclosed charge divided by ϵ_0.

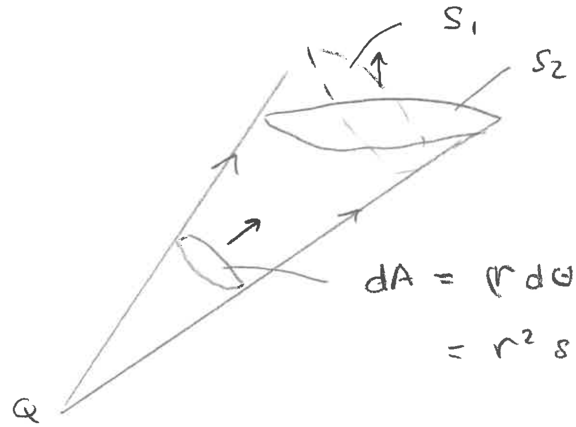
We had 3 examples where integrating over a closed surface

$$\int \bar{E} \cdot \hat{n}\, dA = \frac{Q}{\epsilon_0}$$

where Q is the charge enclosed by the surface and $\hat{n}$ is an outward directed normal.

This is actually a general result called Gauss' law.

To understand it consider a point charge



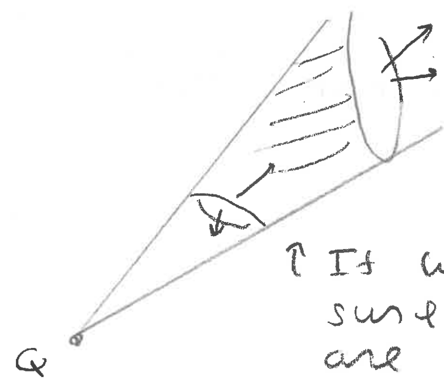
$$dA = (r\,d\theta)(r\sin\theta\, d\phi)$$
$$= r^2 \sin\theta\, d\theta\, d\phi$$

① $\bar{E} = kQ\frac{\hat{r}}{r^2}$ $\quad \bar{E}\cdot\hat{n}\, dA = kQ\left(\frac{r^2}{r^2}\right)\sin\theta\, d\theta\, d\phi$

note the r^2 cancel.

② in terms of our fluid model the # of field lines (flux) going through S_1 and S_2 is the same

③



↑ If we consider this surface the normals are in the opposite directions so the contributions cancel.

$$\therefore \int \bar{E}\cdot\hat{n}\, dA = q/\epsilon_0$$

for a surface surrounding a point charge, (the net # of flux lines crossing the surface is the same for any closed surface.

If there is no charge in the volum bounded by the surface, then

$$\int \bar{E}\cdot\hat{n}\, dA = 0$$

For systems of charges

$$\bar{E} = \sum \bar{E}_i \qquad (\bar{E}_i = \text{field due to } i^{th} \text{ charge})$$

$$\int \bar{E}\cdot\hat{n}\, dA = \sum \int \bar{E}_i\cdot\hat{n}\, dA = \sum \frac{q_i}{\epsilon_0}$$ for charges contained in the volume

This result is called Gauss law

$$\oint \bar{E} \cdot \hat{n} \, dA = \frac{Q}{\epsilon_0}$$

$\oint dA$ means integrate over a closed surface.