In spherical coordinates dA for the surface of a sphere of radius R is

$$dA = (R\,d\theta)(R\sin\theta\,d\phi) \qquad \hat{n} = \hat{r} = \frac{\bar{r}}{r}$$

$$\bar{E} = kq\frac{\hat{r}}{R^2}$$

$$\bar{E}\cdot\hat{n}\,dA = kq\frac{\hat{r}}{R^2}\cdot\hat{r}\cdot R^2\sin\theta\,d\theta\,d\phi$$

note the factors of R^2 cancel; $\hat{r}\cdot\hat{r} = 1$ because they are unit vectors.

$$\bar{E}\cdot\hat{n}\,dA = kq\sin\theta\,d\theta\,d\phi$$

* note this in independent of R

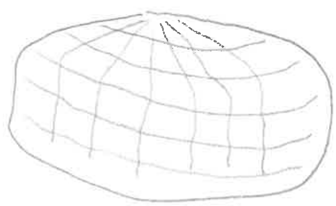
* any surface that has the same number of field lines going through has the same value of $\bar{E}\cdot\hat{n}\,dA = d\Phi$

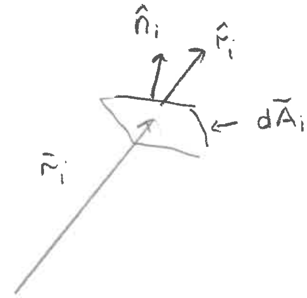
Lecture 7

Gauss' law

electric flux

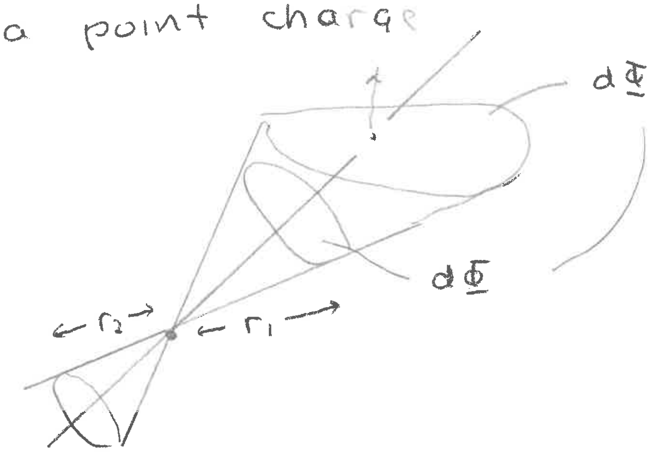
$$\Phi = \lim \sum dA_i \, \hat{n}_i \cdot \bar{E}(\bar{r}_i)$$





It is useful to think of flux like the amount of rain that goes through a surface. In the same way the electric flux is like the number of flux lines through a surface.

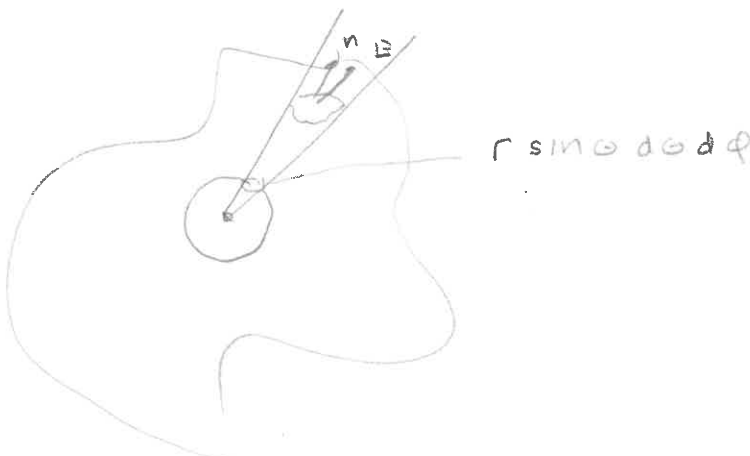
Consider a point charge



both are the same

$\bar{E}_1 = kq \dfrac{\hat{r}}{r_1^2}$ $\qquad \bar{E}_2 = kq \dfrac{\hat{r}}{r_2^2}$

For a point charge at the origin



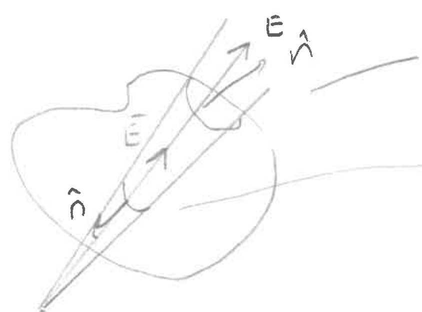
From this picture

$\oint \bar{E}\cdot\hat{n}\, dA$ around an arbitrary closed surface containing a charge q

is the same as

$\oint \bar{E}\cdot\hat{n}\, dA$ = sphere of radius R $= kq \cdot 4\pi = \frac{q}{\epsilon_0}$

alos note



$d\Phi = \sin\theta\, d\theta\, d\phi\, kq\, \hat{r}\cdot\hat{r}$

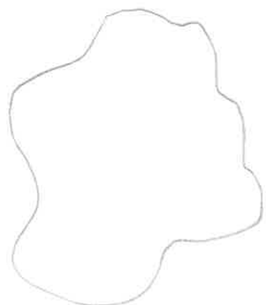
$d\Phi = \sin\theta\, d\theta\, d\phi\, kq\, \hat{r}\cdot(-\hat{r})$

here normal is into sphere

If there is no charge enclosed by the surface

$d\Phi_{in} + d\Phi_{out} = 0$

Putting these things together for a single point charge



$$\oint \bar{E}\cdot\hat{n}\, dA = \begin{cases} q/\epsilon_0 & q \text{ in volume} \\ 0 & q \text{ not in volume} \end{cases}$$

Next we use the superposition principle

$$\bar{E} = \sum_{i=1}^{N} k q_i \frac{\bar{r}-\bar{r}_i}{|\bar{r}-\bar{r}_i|^3}$$

$$\oint \bar{E}\cdot\hat{n}\, dA = \sum_{i=1}^{N} \oint E_i\cdot\hat{n}_i\, dA_i = 4\pi k\, Q_s = \frac{Q_s}{\epsilon_0}$$

where Q_s = is the sum of the charges enclosed by the surface. This result is called Gauss Law

$$\boxed{\oint_S \bar{E}\cdot\hat{n}\, dA = \frac{Q_s}{\epsilon_0}}$$

where S is a closed surface, the convention is that the normal points outside of the closed surface example 1

consider an electric field of the form $\bar{E} = E\hat{r}$ that is constant on a surface of radius r

$$\oint \bar{E}\cdot\hat{n}\, dA = \bar{E}\cdot\int_0^{\pi} r\, d\theta \int_0^{2\pi} r\sin\theta\, d\varphi = \frac{q}{\epsilon_0} = 4\pi k q$$

$$= E\, r^2\, 4\pi$$

this gives

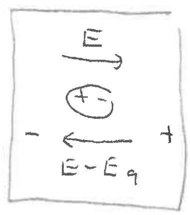
$$\boxed{E = k\frac{q}{r^2}\hat{r}}$$

which is Coulombs law for a point charge.

This shows that Gauss' law is just a different way of expressing coulombs law.

Applications

Electric field inside of a conductor



If there is an electric field in a conductor there will be a force on the charges - since they are free to move in a conductor they will move to the surface and reduce the field in the conductor.

This will continue until there are no more forces on the charges in the interior.

∴ There is no electric field in the interior of an electric conductor

** In general it will take a small amount of time for the charges to move to the surface

The excess charge on a conductor resides on the surface of the conductor consider

$E = \frac{kq}{r^2}$

conductor (uncharged)

Gauss Law $E = k\frac{q}{r^2}$

$0 = k\frac{q + q_{inner\ surf}}{r^2}$

this means the charge on the inner surface is $-q$

the charge on the outer surface is q

If the conductor is also charged gauss law tells us

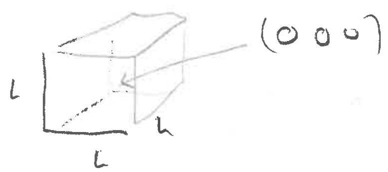
(1) the charge on the inner surface of the conductor is $-q$ (otherwise there will be a field inside of the conductor)

(2) the charge on the outer surface is $q + q_c$ — again by gauss law

∴ all of the excess charge is on the outer surface of the conductor example 2

consider an electric field of the form $\bar{E} = 2x\hat{x} = (2x, 0, 0)$

compute the charge in a cube with



$$\frac{q}{\epsilon_0} = \Phi_D = \sum \Phi \text{ on each plane}$$

$$\Phi_1 = \int_0^L dx \int_0^L dy\ \bar{E}(x\,y\,0)\cdot(-\hat{z}) +$$
$$\int_0^L dx \int_0^L dy\ \bar{E}(x\,y\,L)\cdot(\hat{z}) +$$
$$\int_0^L dy \int_0^L dz\ \bar{E}(0\,y\,z)\cdot(-\hat{x}) +$$
$$\int_0^L dy \int_0^L dz\ \bar{E}(L\,y\,z)\cdot(\hat{x}) +$$
$$\int_0^L dz \int_0^L dx\ \bar{E}(x\,0\,z)\cdot(-\hat{y}) +$$
$$\int_0^L dz \int_0^L dx\ \bar{E}(x\,L\,z)\cdot(\hat{y})$$

since $\bar{E}$ is in the $\hat{x}$ direction $\hat{x}\cdot\hat{y} = \hat{x}\cdot\hat{z} = 0$

the flux through the first 2 and last 2 surfaces vanish.

$$\bar{E}(0\ y\ z) = (0\ 0\ 0)$$
$$\bar{E}(L, y\ z) = (uL, 0, 0)$$

$$\therefore \oint \bar{E}\cdot\hat{n}\,dA = uL\int_0^L dy\int_0^L dz = uL^3 = \frac{q}{\epsilon_0}$$

$$\boxed{q = \epsilon_0 u L^3}$$

This gives the total excess charge in the cube with the field

$$\bar{E} = (ux, 0, 0)$$

<u>example</u> charged conducting plane (infinite

by summetry the field is independent of x, y

If the surface charge density is σ – then half of the charge will be on the upper surface, and half will be on the lower surface

flux on top $E \cdot A$

no flux on sides

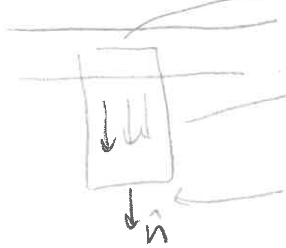
flux on bottom 0 (inside conductor

normal parallel to field on top

$$E \cdot A = \frac{q_s}{\epsilon_0} = \frac{\sigma}{2\epsilon_0} A$$

$$E = \frac{\sigma}{2\epsilon_0} \hat{z}$$

below the plane



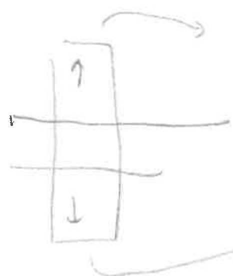
no flux here

no flux here

$\int E \cdot n \, dA = E \cdot A$

$$E \cdot A = \frac{q_s}{\epsilon_0} = \frac{\sigma}{2\epsilon_0} A \rightarrow$$

$$E = \frac{\sigma}{2\epsilon_0} (-\hat{z})$$



$\vec{E}\cdot\hat{n}\,dA = EA$

$\vec{E}\cdot\hat{n}\,dA = EA$

$$2EA = \frac{Q_s}{\epsilon_0} = \frac{\sigma A}{\epsilon_0} \Rightarrow E = \frac{\sigma}{2\epsilon_0}$$

so we see everything is consistent.

return to cylinder - infinite line charge

by symmetry - field is radial - independent of z

$$\oint \vec{E}\cdot\hat{n}\,dA = h\cdot 2\pi R\cdot E = \lambda h/\epsilon_0$$

$$\boxed{\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}\,\hat{r}}$$

This is completely consistent with our calculations based on Coulombs law, but much simpler.

consider $\rho = \rho(\sqrt{x^2+y^2})$ indep of z)
this is an infinite cylindrically symmetric charge distribution

By symmetry $\bar{E}$ is radial $\left(\frac{x, y, 0}{\sqrt{x^2+y^2}}\right)$

$$E 2\pi r h = \int_0^h dz \int_0^r r' dr' \int_0^{2\pi} d\phi \, \rho(r')$$

$$2\pi r h E = 2\pi h \int_0^r r' \rho(r') dr' / \epsilon_0$$

$$\bar{E} = \frac{1}{r\epsilon_0} \int_0^r r' \rho(r') dr' \quad \hat{r}$$

for $\rho(r') = \rho_0 r'$

$$E = \frac{\rho_0}{r\epsilon_0} \int_0^r r'^2 dr' = \frac{\rho_0}{r\epsilon_0} \frac{r^3}{3} = \frac{r^2 \rho_0}{3\epsilon_0}$$

Field inside a uniformly charged sphere with charge density ρ

by symmetry the field is radial

$$\Phi = \int \vec{E}\cdot\hat{n}\, dA = 4\pi r^2 E = \frac{q_s}{\epsilon_0} =$$

$$\int_0^r \rho r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi / \epsilon_0 =$$

$$\frac{4\pi}{\epsilon_0} \rho \frac{r^3}{3}$$

$$\boxed{E = \frac{\rho r}{3\epsilon_0} \hat{r}}$$

in this case the field increases with r

consider the case $\rho(r) = \rho_0 / r^2$

$$\Phi = 4\pi r^2 E = 4\pi \int_0^r \rho r'^2 dr' = 4\pi\rho_0 \int_0^r \frac{r'^2}{r'^2} dr' / \epsilon_0$$

$$= 4\pi \rho_0 r / \epsilon_0$$

$$\boxed{E = \frac{\rho_0}{\epsilon_0} \cdot r \hat{r}}$$

Flux - point charge at center of cube with edges of length 2L

x: $-L \to L$

y: $-L \to L$

z: $-L \to L$

gauss law

$$\oint \bar{E}\cdot\hat{n}\, dA = q/\epsilon_0$$

The integral is the sum of the flux over 6 sides. By symmetry the flux through each side is the same

$$q/\epsilon_0 = 6 \times \Phi_{1\ side}$$

consider the top - coordinates

(x, y, L)

normal $\hat{z}$

$$\bar{E} = \frac{q}{4\pi\epsilon_0}\frac{(x, y, L)}{(x^2+y^2+L^2)^{3/2}}$$

$$\bar{E}\cdot\hat{n}\, dA = \frac{q}{4\pi\epsilon_0}\cdot\frac{L}{(x^2+y^2+L^2)^{3/2}}$$

$$\int \bar{E}\cdot\hat{n}\, dA = \int_0^L dx \int_0^L dy \frac{q}{4\pi\epsilon_0}\frac{L}{(x^2+y^2+L^2)^{3/2}} = \frac{q}{6\epsilon_0}$$

$$\frac{4\pi}{6L} = \boxed{\int_0^L dx \int_0^L dy \frac{1}{(x^2+y^2+L^2)^{3/2}} = \frac{2}{3}\frac{\pi}{L}}$$

here we used Gauss law to evaluate the integral

$$\boxed{\int_0^L dx \int_0^L dy \, \frac{1}{(x^2+y^2+L^2)^{3/2}} = \frac{2}{3}\frac{\pi}{L}}$$

we can also calculate the flux through each side

$$\boxed{\Phi_s = \frac{1}{6}\Phi_T = \frac{1}{6}\frac{q}{\epsilon_0}}$$