Lecture 8

Applications of Gauss Law

$$\boxed{\oint_S \bar{E} \cdot \hat{n}\, dA = \frac{q}{\epsilon_0}} \qquad k = \frac{1}{4\pi\epsilon_0}$$

$$\epsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{Nm^2}$$

here S is a closed surface

q is the net charge enclosed by the surface

$\hat{n}$ is an <u>outward</u> pointing unit normal vector at each point on the surface

dA is a differential area ($dx\,dy$, $r^2 \sin\theta\, d\theta\, d\phi$, ---)

Gauss' Law is the equation in the box.

We derived this by starting with a single point charge and then used the superposition principle

An important result that can be used in applying Gauss law is that there is no static electric field in the interior of a conductor

(static means that we wait until the charges have stopped moving - which causes the field to change.)

① applications with spherical geometries

for these applications the symmetry considerations imply that (1) the magnitude of the field is independent of the angle and (2) the direction of the field is parallel or antiparallel to the radial direction

* point charge at origin

calculate flux through a sphere of radius r centered on the charge

$$\Phi = \int \bar{E}\cdot\hat{n}\, dA = \underbrace{E(r)\hat{r}}_{\bar{E}} \cdot \underbrace{\hat{r}}_{\hat{n}} \underbrace{(4\pi r^2)}_{A} = q/\epsilon_0$$

this give

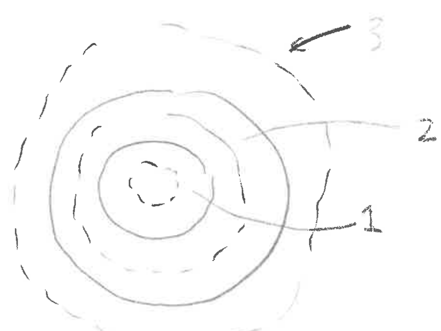
$$\boxed{\bar{E}(r) = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}}$$

using

$$\bar{F} = q'\bar{E} = \frac{q'q}{4\pi\epsilon_0}\cdot\frac{\hat{r}}{r^2} = k\frac{q'q}{r^2}\hat{r}$$

which is just Coulombs law for a point charge. Since we derived Gauss' law from Coulombs law we see that they are equivalent

② charged conducting spherical shell



consider the 3 spherical surfaces 1, 2, 3 since there is no charge in the centre

$$\int_{S_1} \bar{E}\cdot\hat{n}\, dA = \bar{E}(r)\hat{r}\, 4\pi r^2 = \frac{0}{\epsilon_0}$$

$$\bar{E}(r) = 0$$

the field in the interior must vanish

next consider

$$\int_{S_2} \bar{E}\cdot\hat{n}\, dA = \bar{E}(r)\hat{r}\hat{r}\, 4\pi r^2 = \frac{Q}{\epsilon_0}$$

since S_2 is in the interior of a conductor, the field vanishes.

this means that there is no charge on the <u>inner</u> surface of the conductor.

the integral over S_3 gives

$$\int_{S_3} \bar{E}\cdot\hat{n}\, dA = \bar{E}(r)\, 4\pi r^2 = \frac{Q}{\epsilon_0}$$

in this case, since the conductor is charged - all of the charge is uniformly distributed over the outer surface.

$\therefore$ $\bar{E} = \frac{kq}{r^2}\hat{r}$ outside of the sphere

For the same geometry, now assume there is a charge in the center q_1 and the charge on the conductor is q_2

Then we have

$$\oint_{S_1} \bar{E}\cdot\hat{n}\,dA = E(r)4\pi r^2 = \frac{q_1}{\epsilon_0}$$

$$E(r) = \frac{q_1}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$$

$$\oint_{S_2} \bar{E}\cdot\hat{n}\,dA = 0\cdot 4\pi r^2 = \frac{q_1 + q_{inner}}{\epsilon_0}$$

since the field is 0,

$$q_{inner} = -q_1$$

$$\oint_{S_3} \bar{E}\cdot\hat{n}\,dA = E\,4\pi r^2 = \frac{q_1+q_2}{\epsilon_0}$$

$$E(r) = \frac{(q_1+q_2)}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$$

in this case outside of the conductor the field is the same as the field due to a charge q_1+q_2 at the origin we can also treat radially symmetric charge densities

$$\rho(r) = \begin{cases} \rho_0 & r < R \\ 0 & r > R \end{cases}$$

(uniformly charged sphere

$$\oint \bar{E} \cdot \hat{n} \, dA = E(r) 4\pi r^2$$

$$= \int \frac{\rho(r)\, dV}{\epsilon_0} =$$

$$= \begin{cases} \int_0^r \frac{\rho_0}{\epsilon_0} dr \int_0^\pi r \, d\theta \int_0^{2\pi} r \sin\theta \, d\phi & r < R \\ \int_0^R \frac{\rho_0}{\epsilon_0} dr \int_0^\pi r \, d\theta \int_0^{2\pi} r \sin\theta \, d\phi & r > R \end{cases}$$

$$= \begin{cases} \int_0^r r^2 dr \, 4\pi \frac{\rho_0}{\epsilon_0} = \frac{4}{3}\pi r^3 \frac{\rho_0}{\epsilon_0} & r < R \\ \int_0^R r^2 dr \, 4\pi \frac{\rho_0}{\epsilon} = \frac{4}{3}\pi R^3 \frac{\rho_0}{\epsilon_0} & r > R \end{cases}$$

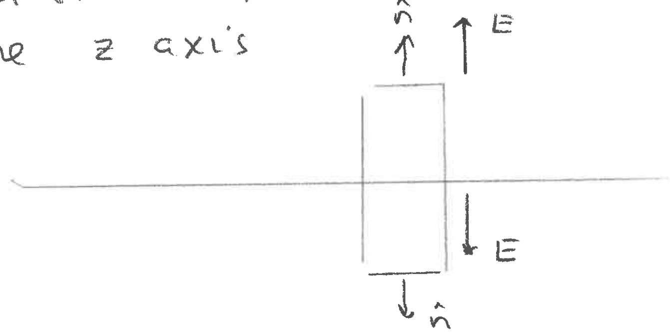
$$E = \begin{cases} \frac{1}{4\pi r^2} \frac{4}{3}\pi r^3 \frac{\rho_0}{\epsilon_0} = \frac{\rho_0 r}{3\epsilon_0} \hat{r} = \frac{\rho_0 \bar{r}}{3\epsilon_0} & r < R \\ \frac{1}{4\pi r^2} \frac{4}{3}\pi R^3 \frac{\rho_0}{\epsilon_0} = \frac{\rho_0}{3\epsilon_0} \frac{R^3 \hat{r}}{r^2} = \frac{Q}{\epsilon_0} \frac{\hat{r}}{r^2} & r > R \end{cases}$$

plane symmetry

consider a uniformly charged plane with charge density σ.
(assume the plane is the x-y plane)

By symmetry

① the field in independent of x, y
② the field is parallel or antiparallel to the z axis



* no flux on vertical faces
* flux on top and bottom

$$(E\cdot\hat{z})\cdot\hat{z}\,A + E(-\hat{z})\cdot(-\hat{z})\cdot A = \sigma A/\epsilon_0$$

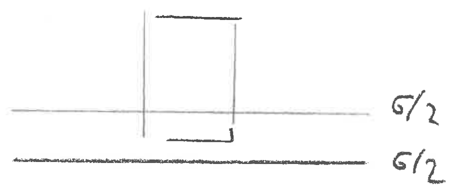
(using Gauss law)

$$2EA = \sigma A/\epsilon_0$$

dividing by A

$$E = \sigma/2\epsilon_0$$

The result is independent of z - this is the same result we obtained using Gauss law charged conducting plane



The symmetry conditions are the same

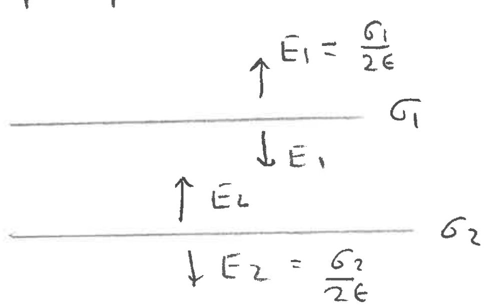
- no flux on vertical edges
- no flux in conductor

$$\Phi = (E \cdot \hat{x}) \cdot (A \hat{x}) = \frac{\sigma}{2} A / \epsilon_0$$

$$E = \frac{\sigma}{2\epsilon_0}$$

gives the same field

case of 2 planes - different densities
use the superposition principle



1 $E = \frac{\sigma_1 + \sigma_2}{2\epsilon} \hat{z}$

2 $E = \frac{\sigma_2 - \sigma_1}{2\epsilon} \hat{z}$

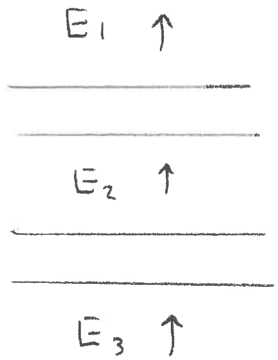
3 $E = -\frac{\sigma_1 + \sigma_2}{2\epsilon} \hat{z}$

If $\sigma_1 = -\sigma_2 \equiv \sigma$ then

(1) there is no field in regions 1, 3

(2) field in region 2 is $\frac{2\sigma_2}{2\epsilon_0} \hat{z} = \frac{\sigma_2}{\epsilon_0} \hat{z} = -\frac{\sigma}{\epsilon_0} \hat{z}$

consider 2 conducting planes



$\sigma_1 = \sigma_{1\,TOP} + \sigma_{1\,BOT}$

$\sigma_2 = \sigma_{2\,top} + \sigma_{2\,Bot}$

* we know there is no field in the conductor

* from last problem (superposition)

$$E_1 = (\sigma_1 + \sigma_2)/2\epsilon_0 \ \hat{z}$$

$$E_3 = -(\sigma_1 + \sigma_2)/2\epsilon_0 \ \hat{z} \qquad \text{(normals opposite)}$$

$$E_2 = \frac{\sigma_2 - \sigma_1}{2\epsilon_0} \ \hat{z}$$

× from there, now that we know the field we can find

$$E_1 A = \sigma_{1T} A/\epsilon_0$$

$$\sigma_{1T} = \epsilon_0 E_1 = \frac{\sigma_1}{2} + \frac{\sigma_2}{2}$$

$$\sigma_{1B} = -\epsilon_0 E_2 = \frac{\sigma_2}{2} - \frac{\sigma_1}{2}$$

$$\sigma_{2T} = \epsilon E_2 = -\sigma_{1B} = \frac{\sigma_1 - \sigma_2}{2}$$

$$\sigma_{2B} = -\epsilon_0 E_3 = -(-)\left(\frac{\sigma_1 + \sigma_2}{2}\right)$$

for opposite charges $\sigma_1 = -\sigma_2 = \sigma$

$$\sigma_{1T} = \sigma_{2B} = 0$$

$$\sigma_{1B} = \sigma_2 = -\sigma_1 = -\sigma_{2T}$$

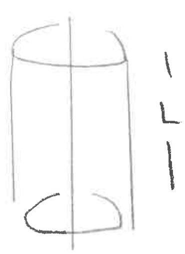
so if the charge densities are equal and opposite all of the charges are on the inner surface of the conductors.

cylinders

infinite cylindrically symmetric charge distributions

By symmetry the field is independent of z (axis of symmetry) - field must be radial

* line charge λ = charge / length



$$\Phi = \Phi_{TOP} + \Phi_{BOT} + \Phi_{SIDES} = Q/\epsilon_0$$

$$Q/\epsilon_0 = \frac{\lambda L}{\epsilon_0} = \frac{\text{charge in cylinder}}{\epsilon_0}$$

$$\Phi_{TOP} = \Phi_{BOT} = 0 \quad \hat{n} \perp E$$

$$\Phi_{side} = E\, 2\pi r L \quad \hat{n} \parallel E$$

$$E \cdot 2\pi r \cancel{L} = \frac{\lambda \cancel{L}}{\epsilon_0}$$

$$E = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{r} \rightarrow \boxed{\vec{E} = \frac{\lambda}{2\pi\epsilon_0}\frac{\hat{r}}{r}}$$

which is the same result we got using coulomb's law - this time with no integration case 2 $\rho = \rho(r)$ $r^2 = x^2 + y^2$

$$\Phi = E(r) \cdot 2\pi r L = \frac{Q}{\epsilon_0} = \int_0^L dz \int_0^r \rho(r')\,dr' \int_0^{2\pi} r'\,d\phi \,/\, \epsilon_0$$

$$= \frac{L}{\epsilon_0} \cdot 2\pi \int_0^r r' \rho(r')\,dr'$$

$$\boxed{\bar{E} = \frac{1}{r\epsilon_0} \int_0^r r' \rho(r')\,dr' \;\hat{r}}$$

if $\rho(r) = \frac{c}{r}$ $\bar{E} = \frac{1}{r} \cdot \frac{1}{\epsilon_0} \cdot c r \hat{r} = \frac{c}{\epsilon_0} \hat{r}$

we can use different $\rho(r)$

Doing Integrals

consider a point charge q at the center of a cube of side $2L$

① The total flux is $\frac{Q}{\epsilon_0}$.

② By symmetry the flux through each face is the same

$$\Phi_{FACE} = \frac{1}{6} \Phi_{TOTAL} = \frac{Q}{6\epsilon_0}$$

for the upper face

$$\frac{Q}{6\epsilon_0} = \int_{-L}^{L} dx \int_{-L}^{L} dy \; \bar{E}(x, y, L) \cdot \hat{z}$$

$$= kQ \int_{-L}^{L} dx \int_{-L}^{L} dy \; \frac{L}{(x^2+y^2+L^2)^{3/2}}$$

$$= \frac{QL}{4\pi\epsilon_0} \int_{-L}^{L} dx \int_{-L}^{L} dy \; \frac{1}{(x^2+y^2+L^2)^{3/2}}$$

This gives - canceling Q/ϵ_0

$$\boxed{\frac{4\pi}{6L} = \frac{2\pi}{3L} = \int_{-L}^{L} dx \int_{-L}^{L} dy \; \frac{1}{(x^2+y^2+L^2)^{3/2}}}$$

Electrostatic potential

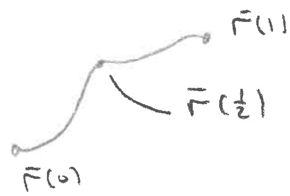
The work done in moving a charge Q particle against a field $\bar{E}$

$$dW = -\bar{F}\cdot d\bar{r} = -Q\bar{E}\cdot d\bar{r}$$

If $\bar{E}$ is due to a point charge at at the origin

$$E(\bar{r}) = kq\,\frac{\bar{r}}{r^3}$$

consider a path $\bar{r}(s)$ $\quad s: 0 \to 1$



$$\bar{r}(s) = (x(s), y(s), z(s))$$

$$d\bar{r} = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) ds$$

$$dW = -Q(kq)\,\frac{(x, y, z)}{r^3}\cdot\left(\frac{dx}{dx}, \frac{dy}{ds}, \frac{dz}{ds}\right) ds$$

$$= -Q(kq)\,\frac{x\frac{dx}{ds} + y\frac{dy}{ds} + z\frac{dz}{ds}}{(x^2+y^2+z^2)^{3/2}}\,ds$$

$$= -kQq\,\underbrace{\frac{d}{ds}\left(\frac{-1}{\sqrt{x^2+y^2+z^2}}\right)}_{-\left(-\frac{1}{2}\right)\frac{\left(2x\frac{dx}{ds}+2y\frac{dy}{ds}+2z\frac{dz}{ds}\right)}{(x^2+y^2+z^2)^{3/2}}}\,ds$$

$$= (-1)^2 kQq\,\frac{d}{ds}\,\frac{1}{\sqrt{x^2(s)+y^2(s)+z^2(s)}}\,ds$$

Integrating

$$\int_{\bar{r}(0)}^{\bar{r}(1)} dW = \int_0^1 \frac{dW}{ds}\,ds = kQq\left(\frac{1}{r(1)} - \frac{1}{r(0)}\right)$$

The key observation is that the work done againts the field

(1) is independent of the path - it only depends on the initial and final distance from the charge.

(2) This work increases the potential energy U

$$\Delta U = kQq\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$$

Next consider a field due to many point charges

$$dW = -\bar{F}\cdot d\bar{r} = -Q\bar{E}\cdot d\bar{r}$$

$$= -Q\left(\sum \bar{E}_n(\bar{r})\cdot d\bar{r}\right)$$

$$= kQ\sum q_n\left(\frac{1}{|\bar{r}_f-\bar{r}_n|} - \frac{1}{|\bar{r}_i-\bar{r}_n|}\right)$$

$$\Delta U = kQ\sum q_n\left(\frac{1}{|\bar{r}_f-\bar{r}_n|} - \frac{1}{|\bar{r}_i-\bar{r}_n|}\right)$$

This quantity is independent of the path taken by the particle we let

$$\Delta V = \frac{\Delta U}{Q} = \text{change in the electrostatic potential}$$

$$\boxed{dV = -\bar{E} \cdot d\bar{r}} \quad \text{since} \quad dW = dV$$

since this is independent of path

$$V(\bar{r}) - V(\bar{r}_0) = -\int \bar{E}(\bar{r}) \cdot d\bar{r}$$

along any path between $\bar{r}_0$ and $\bar{r}$

$$\bar{r}(s) : \quad \bar{r}(0) = \bar{r}_0 \qquad \bar{r}(1) = \bar{r}$$

the unit of electrostatic potential is

$$\boxed{1 \text{ Volt} = \frac{1 \text{ Newton} \cdot \text{meter}}{\text{coulomb}}}$$

For a point charge at the origin

$$\boxed{V = \frac{kq}{r}} \quad \text{potential for a point charge}$$

choosing $V(\infty) = 0$