Lecture 8

Applications of Gauss Law

\[ \oint_S \mathbf{E} \cdot \hat{n} \, dA = \frac{q}{\epsilon_0} \]

\[ R = \frac{1}{4\pi \epsilon_0} \]

\[ \epsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{Nm^2} \]

Here \( S \) is a closed surface

\( q \) is the net charge enclosed by the surface

\( \hat{n} \) is an outward pointing unit normal vector at each point on the surface

\( dA \) is a differential area

\( (dxdy, r^2 \sin\theta d\theta d\phi, \ldots) \)

Gauss' Law is the equation in the box.

We derived this by starting with a single point charge and then used the superposition principle.
An important result that can be used in applying Gauss’s law is that there is no static electric field in the interior of a conductor.

(static means that we wait until the changes have stopped moving—which causes the field to change.)

1. Applications with spherical geometries

For these applications the symmetry considerations imply that (1) the magnitude of the field is independent of angle and (2) the direction of the field is parallel or antiparallel to the radial direction.

* Point charge at origin
  * Calculate flux through a sphere of radius r centered on the charge
\[ \Phi = \int \vec{E} \cdot \hat{n} \, dA = \vec{E} \cdot \hat{n} \, \hat{A} \left[ \left( 0 \right) \, \hat{r} \right] = \frac{q}{4 \pi \varepsilon_0 r^2} \hat{r} \]

This gives

\[ \vec{E}(r) = \frac{q}{4 \pi \varepsilon_0 r^2} \hat{r} \]

using

\[ \vec{E} = q' \vec{E} = \frac{q' q}{4 \pi \varepsilon_0 r^2} \hat{r} = k \frac{q' q}{r^2} \hat{r} \]

which is just Coulomb's law for a point charge, since we derived Gauss' law from Coulomb's law we see that they are equivalent

2) changed conducting spherical shell

Consider the 3 spherical surfaces 1, 2, 3
since there is no change in
the center
\[ \int_{S_1} \mathbf{E} \cdot \hat{n} \, dA = \mathbf{E}(r) \hat{n} \cdot 4\pi r^2 = 0 \]
\( \mathbf{E}(r) = 0 \)

the field in the interior must vanish

next consider
\[ \int_{S_2} \mathbf{E} \cdot \hat{n} \, dA = \mathbf{E}(r) \hat{n} \cdot 4\pi r^2 = 0 \]

since \( S_2 \) is in the interior of a
conductor, the field vanishes.

this means that there is
no change on the inner
surface of the conductor.

the integral over \( S_3 \) gives
\[ \int_{S_3} \mathbf{E} \cdot \hat{n} \, dA = \mathbf{E}(r) \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0} \]

in this case, since the conductor
is charged - all of the charge
is uniformly distributed over
the outer surface.

\( \Rightarrow \mathbf{E} = \frac{kQ}{r^2} \) outside of the sphere
For the same geometry, now assume there is a change in the center $q_1$ and the change on the conductor is $q_2$.

Then we have

\[
\oint_{S_1} \mathbf{E} \cdot \mathbf{n} \, dA = E(r) \, 4\pi \, r^2 = \frac{q_1}{\varepsilon_0}
\]

\[
E(r) = \frac{q_1}{4\pi \varepsilon_0 \, r^2}
\]

\[
\oint_{S_2} \mathbf{E} \cdot \mathbf{n} \, dA = 0 \cdot 4\pi \, r^2 = \frac{q_1 + q_{\text{inner}}}{\varepsilon_0}
\]

since the field is 0,

$q_{\text{inner}} = -q_1$.

\[
\oint_{S_3} \mathbf{E} \cdot \mathbf{n} \, dA = E \cdot 4\pi \, r^2 = \frac{q_1 + q_2}{\varepsilon_0}
\]

\[
E(r) = \frac{(q_1 + q_2)}{4\pi \varepsilon_0} \frac{1}{r^2}
\]

In this case outside of the conductor the field is the same as the field due to a charge $q_1 + q_2$ at the origin.
We can also treat radially symmetric charge densities

\[ p(r) = \begin{cases} p_0 & r < R \\ 0 & r > R \end{cases} \]

(Uniformly charged sphere)

\[ \oint E \cdot n \, dA = E(r) \cdot 4\pi r^2 \]

\[ = \int \frac{p(r)}{\varepsilon_0} \, dV = \]

\[ = \begin{cases} \int_0^R \frac{p_0}{\varepsilon_0} \, dr \int_0^{\pi} d\theta \int_0^{2\pi} \sin \theta \, d\phi & r < R \\ \int_0^R \frac{p_0}{\varepsilon_0} \, dr \int_0^{\pi} d\theta \int_0^{2\pi} \sin \theta \, d\phi & r > R \end{cases} \]

\[ = \begin{cases} \int_0^R r^2 \, dr \cdot 4\pi \frac{p_0}{\varepsilon_0} = \frac{4}{3} \pi r^3 \frac{p_0}{\varepsilon_0} & r < R \\ \int_0^R r^2 \, dr \cdot 4\pi \frac{p_0}{\varepsilon_0} = \frac{4}{3} \pi R^3 \frac{p_0}{\varepsilon_0} & r > R \end{cases} \]

\[ E = \begin{cases} \frac{1}{4\pi r^2} \cdot \frac{4}{3} \pi r^3 \frac{p_0}{\varepsilon_0} = \frac{p_0}{3\varepsilon_0} \frac{\hat{r}}{r} = \frac{p_0}{3\varepsilon_0} \frac{\hat{r}}{r^2} & r < R \\ \frac{1}{4\pi r^2} \cdot \frac{4}{3} \pi R^3 \frac{p_0}{\varepsilon_0} = \frac{p_0}{3\varepsilon_0} \frac{R^3 \hat{r}}{r^2} = \frac{0}{\varepsilon_0} \frac{\hat{r}}{r^2} & r > R \end{cases} \]
plane symmetry

consider a uniformly charged plane with charge density \( \sigma \).
(assume the plane is the x-y plane)

By symmetry

1. the field in independent of x, y
2. the field is parallel or antiparallel to the z axis

\[ \mathbf{E} \]

* no flux on vertical faces
* flux on top and bottom

\[ (\mathbf{E} \cdot \hat{z}) \cdot \hat{z} A + (\mathbf{E} \cdot \hat{-z}) \cdot (-\hat{z}) A = \sigma A / \varepsilon_0 \]

(using gauss law)

\[ 2EA = \sigma A / \varepsilon_0 \]

dividing by \( A \)

\[ E = \sigma / 2 \varepsilon_0 \]

the result is independent of \( z \).
this is the same result we obtained using gauss law
changed conducting plane

\[
\begin{array}{c}
\frac{\sigma}{2} \\
\frac{\sigma}{2}
\end{array}
\]

The symmetry conditions are the same
- no flux on vertical edges
- no flux in conductors

\[ \Phi = (E, \hat{z}), (A \hat{z}) = \frac{\sigma}{2} A / \varepsilon_0 \]
\[ E = \frac{\sigma}{2 \varepsilon_0} \]

gives the same field

case of 2 planes - different densities use the superposition principle

\[
\begin{array}{ccc}
\uparrow E_1 = \frac{\sigma_1}{2 \varepsilon_0} & 1 & E = \frac{\sigma_1 + \sigma_2}{2 \varepsilon_0} \hat{z} \\
\downarrow E_2 = \frac{\sigma_2}{2 \varepsilon_0} & 2 & E = \frac{\sigma_2 - \sigma_1}{2 \varepsilon_0} \hat{z} \\
\downarrow E_2 = \frac{\sigma_2}{2 \varepsilon_0} & 3 & E = -\frac{\sigma_1 + \sigma_2}{2 \varepsilon_0} \hat{z}
\end{array}
\]

If \( \sigma_1 = -\sigma_2 = \sigma \) then

1) there is no field in regions 1, 3
2) field in region 2 is \( \frac{2 \sigma}{2 \varepsilon} \hat{z} = \frac{\sigma_2}{\varepsilon_0} \hat{z} = -\frac{\sigma}{\varepsilon} \hat{z} \)
Consider 2 conducting planes

\[ \mathbf{E}_1 \uparrow \]
\[ \mathbf{E}_2 \uparrow \]
\[ \mathbf{E}_3 \uparrow \]

\[ G_1 = G_{1\text{top}} + G_{1\text{bot}} \]
\[ G_2 = G_{2\text{top}} + G_{2\text{bot}} \]

* We know there is no field in the conductors

* From last problem (superposition)

\[ \mathbf{E}_1 = (G_1 + G_2) / 2 \varepsilon_0 \hat{z} \]
\[ \mathbf{E}_3 = - (G_1 + G_2) / 2 \varepsilon_0 \hat{z} \quad (\text{normals opposite}) \]
\[ \mathbf{E}_2 = \frac{G_2 - G_1}{2 \varepsilon_0} \hat{z} \]

* From these, now that we know the field we can find

\[ \mathbf{E}_1 \mathbf{A} = G_{1\text{top}} \mathbf{A} / \varepsilon_0 \]
\[ G_{1\text{top}} = \mathbf{E}_1 \mathbf{E}_1 = \frac{G_1 + G_2}{2} \]
\[ G_{1\text{bot}} = - \mathbf{E}_1 \mathbf{E}_2 = \frac{G_2 - G_1}{2} \]
\[ G_{2\text{top}} = \mathbf{E}_2 \mathbf{E}_2 = - G_{1\text{bot}} = \frac{G_1 - G_2}{2} \]
\[ G_{2\text{bot}} = - \mathbf{E}_2 \mathbf{E}_3 = -(\cdot) \left( \frac{G_1 + G_2}{2} \right) \]

For opposite charges \( G_1 = - G_2 = 6 \)

\[ G_{1\text{top}} = G_{2\text{top}} = 0 \]
\[ G_{1\text{bot}} = G_2 = - G_1 = - G_{2\text{top}} \]
so if the charge densities are equal and opposite all of the charges are on the inner surface of the conductors.

Cylinders
Infinite cylindrically symmetric charge distributions

By symmetry the field is independent of $z$ (axis of symmetry) - field must be radial.

* Line charge $\lambda = \text{change in length}$

\[ \Phi = \Phi_{\text{top}} + \Phi_{\text{bot}} + \Phi_{\text{sides}} = \frac{Q}{\varepsilon_0} \]

\[ \frac{Q}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0} = \frac{\text{change in cylinder}}{\varepsilon_0} \]

\[ \Phi_{\text{top}} = \Phi_{\text{bot}} = 0 \quad \hat{n} \perp \vec{E} \]

\[ \Phi_{\text{sides}} = \vec{E} \cdot 2\pi r \hat{n} \perp \vec{E} \]

\[ \vec{E} \cdot 2\pi r \hat{n} = \frac{\lambda L}{\varepsilon_0} \]

\[ \vec{E} = \frac{1}{2\pi\varepsilon_0} \hat{n} \rightarrow \left[ \frac{\hat{E}}{2\pi\varepsilon_0} \right] \]

Which is the same result we got using Coulomb law - this time with no integration.
Case 2 \( \rho = \rho(r) \) \( \mathbf{r} = x^2 + y^2 \)

\[
\mathbf{E} = E(r) \cdot 2\pi r L = \frac{Q}{\varepsilon_0} \int_0^L dx \int_0^{2\pi} \rho(r) dr \int_0^{2\pi} d\phi \frac{r}{\varepsilon_0} \int_0^r \rho(r') dr' \\
\mathbf{E} = \frac{1}{\varepsilon_0} \int_0^r \rho(r') dr' \mathbf{r}
\]

If \( \rho(r) = \frac{C}{r} \)

\[
\mathbf{E} = \frac{1}{\varepsilon_0} \cdot \frac{C}{r} \mathbf{r} = \frac{C}{\varepsilon_0} \mathbf{r}
\]

we can use different \( \rho(r) \).

Doing Integrals

Consider a point charge \( Q \) at the center of a cube of side \( 2L \)

1. The total flux is \( \frac{Q}{\varepsilon_0} \).

2. By symmetry, the flux through each face is the same.

\[
\mathbf{E}_{\text{face}} = \frac{1}{6} \mathbf{E}_{\text{total}} = \frac{Q}{6\varepsilon_0}
\]

For the upper face

\[
\frac{Q}{6\varepsilon_0} = \int_{-L}^{L} dx \int_{-L}^{L} dy \ E(x, y, L) \cdot \mathbf{z} = KQ \int_{-L}^{L} dx \int_{-L}^{L} dy \ \frac{L}{(x^2 + y^2 + L^2)^{3/2}}
\]

\[
= \frac{QL}{4\pi\varepsilon_0} \int_{-L}^{L} dx \int_{-L}^{L} dy \ \frac{1}{(x^2 + y^2 + L^2)^{1/2}}
\]
This gives - cancelling $Q/E_0$

\[
\frac{4 \pi l}{6L} = 3L = \int_{-L}^{L} dx \int_{-L}^{L} dy \frac{1}{(x^2 + y'^2 + L^2)^{3/2}}
\]
Electrostatic potential

The work done in moving a charge $Q$ particle against a field $\vec{E}$

$$dW = -\vec{F} \cdot d\vec{r} = -Q \vec{E} \cdot d\vec{r}$$

If $\vec{E}$ is due to a point charge at the origin

$$\vec{E}(\vec{r}) = \frac{Q}{r^2} \hat{r}$$

Consider a path $\vec{r}(s)$: $0 \to 1$

$$\vec{r}(s) = (x(s), y(s), z(s))$$

$$d\vec{r} = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) ds$$

$$dW = -Q \left(\frac{x, y, z}{r^3}\right) \cdot \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) ds$$

$$= -Q \left(\frac{x \frac{dx}{ds} + y \frac{dy}{ds} + z \frac{dz}{ds}}{\sqrt{x^2 + y^2 + z^2}}\right) ds$$

$$= -Q \left(\frac{\sqrt{\frac{x^2 + y^2 + z^2}{\sqrt{x^2 + y^2 + z^2}}}}{x^2 + y^2 + z^2}\right) ds$$

$$= -Q \left(\frac{\sqrt{\frac{x^2 + y^2 + z^2}{\sqrt{x^2 + y^2 + z^2}}}}{x^2 + y^2 + z^2}\right) ds$$

$$= -Q \left(\frac{1}{\sqrt{x^2 + y^2 + z^2}}\right) ds$$

Integrating

$$\int_{\vec{r}(0)}^{\vec{r}(1)} dW = \int_{\vec{r}(0)}^{\vec{r}(1)} \frac{Q}{r(s)} \left(\frac{1}{r(0)} - \frac{1}{r(1)}\right) ds$$
The key observation is that the work done against the field

(1) is independent of the path - it only depends on the initial and final distance from the charge.

(2) This work increases the potential energy $U$

$$\Delta U = kq \left( \frac{1}{r_f} - \frac{1}{r_i} \right)$$

Next consider a field due to many point charges

$$dW = -\mathbf{F} \cdot d\mathbf{r} = -\mathbf{q} \cdot d\mathbf{F}$$

$$= -q \left( 2 \mathbf{E} \cdot d\mathbf{r} \right)$$

$$= kq \sum q_i \left( \frac{1}{|r_f - r_i|} - \frac{1}{|r_i - r_f|} \right)$$

$$\Delta U = kq \sum q_i \left( \frac{1}{|r_f - r_i|} - \frac{1}{|r_i - r_f|} \right)$$

This quantity is independent of the path taken by the particle.
We let
\[ \Delta V = \frac{\Delta U}{q} = \text{change in the electrostatic potential} \]
\[ dV = -\vec{E} \cdot d\vec{r} \quad \text{since} \quad dw = dv \]

since this is independent of path
\[ V(\vec{r}) - V(\vec{r}_0) = -\int \vec{E}(\vec{r}) \cdot d\vec{r} \]

along any path between \( \vec{r}_0 \) and \( \vec{r} \)
\[ \vec{r}(s): \vec{r}(0) = \vec{r}_0, \quad \vec{r}(1) = \vec{r} \]

The unit of electrostatic potential is
\[ 1 \text{ Volt} = \frac{1 \text{ Newton \cdot meter}}{\text{coulomb}} \]

For a point charge at the origin
\[ V = \frac{kq}{r} \]

potential for a point charge
choosing \( V(\infty) = 0 \)