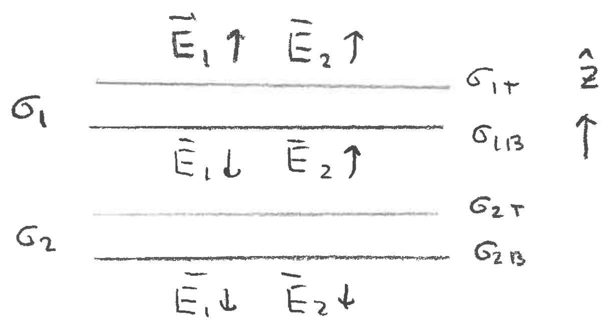
Lecture 9

Last time

$$\Phi = \oint_S \vec{E}(\vec{r}) \cdot \hat{n}(\vec{r})\, dA = \frac{Q_s}{\epsilon_0} \qquad \text{Gauss Law}$$

* do integrals using symmetry
* use no field inside of conductors
* use superposition principle

example 1 2 uniformly charged infinite conducting planes



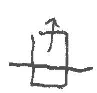
single plane
$$\vec{E} = \frac{\sigma}{2\epsilon_0} \cdot \hat{n}$$

superposition gives

$$\vec{E}_{TOP} = \vec{E}_1 + \vec{E}_2 = \frac{\sigma_1 + \sigma_2}{2\epsilon_0}\hat{z}$$

$$E_{MIDDLE} = \vec{E}_1 + \vec{E}_2 = \frac{\sigma_2 - \sigma_1}{2\epsilon_0}\hat{z}$$

$$E_{BOTTOM} = \vec{E}_1 + \vec{E}_2 = -\frac{\sigma_1 + \sigma_2}{2\epsilon_0}\hat{z}$$

σ conductor  $E \cdot A = \frac{\sigma A}{\epsilon_0}$ $\quad \vec{E} = \frac{\sigma}{\epsilon_0}$ $\quad \sigma = \epsilon_0 E$

$$\sigma_{1T} = \epsilon_0 \frac{\sigma_1 + \sigma_2}{2\epsilon_0} = \frac{1}{2}(\sigma_1 + \sigma_2)$$

$$\sigma_{1B} = \epsilon_0 (-1) \frac{\sigma_2 - \sigma_1}{2\epsilon_0} = \frac{1}{2}(\sigma_1 - \sigma_2)$$

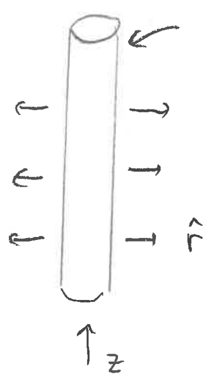
$$\sigma_{2T} = \epsilon_0 (1) \frac{\sigma_2 - \sigma_1}{2\epsilon_0} = \frac{1}{2}(\sigma_2 - \sigma_1) = -\sigma_{1B}$$

$$\sigma_{2B} = \epsilon_0 (-1)^2 \frac{\sigma_1 + \sigma_2}{2\epsilon_0} = \frac{1}{2}(\sigma_1 + \sigma_2)$$

when $\sigma_2 = -\sigma_1 = -\sigma$

$$\boxed{\begin{array}{l} \bar{E}_T = \bar{E}_B = 0 \quad \bar{E}_M = -\frac{\sigma}{\epsilon_0}\hat{z} \\ \sigma_{1T} = \sigma_{2B} = 0 \ ; \quad \sigma_{1B} = \sigma_1 = -\sigma_{2T} \end{array}}$$

Cylindrical Symmetry



uniformly charged infinite cylinder with charge density ρ ; radius R

* by symmetry the field is radially outward $\hat{r}$.
* the normal on the face of a Gaussian cylinder is $\hat{n} = \hat{r}$
* the normal on the top or bottom is $\hat{z}, -\hat{z}$
* by symmetry $\bar{E} = E(r)\hat{r}$ independent of z.

$$\Phi = \oint_{\square} \bar{E}\cdot\hat{n}\, dA = \int_{top} (E(r)\underbrace{\hat{r})\cdot\hat{z}}_{0}\, dA +$$

$$\int_{BOT} (E(r)\underbrace{\hat{r})\cdot(-\hat{z}}_{0})\, dA + \int^{L} dz \int^{2\pi} r\, d\theta\ E(r)\,\hat{r}\cdot\hat{r} =$$

$$E(r)\cdot 2\pi r L = \frac{1}{\epsilon_0} Q(r) =$$

The volume of the cylinder is

$\pi r^2 L$

outside the cylinder the total charge is

$$Q = \pi R^2 L \rho$$

inside - a distance r from the z axis the charge in the gaussian surface is

$$Q = \pi r^2 L \rho$$

$$\therefore \quad \bar{E}(r) = E(r)\hat{r} = \frac{Q(r)}{2\pi r L \epsilon_0} \hat{r}$$

$$= \frac{\hat{r}}{2\pi r L \epsilon_0} \begin{cases} \pi R^2 L \rho & \text{outside} \\ \pi r^2 L \rho & \text{inside} \end{cases}$$

$$= \frac{\hat{r}\rho}{2\epsilon_0} \begin{cases} \frac{R^2}{r} & \text{outside} \\ r & \text{inside} \end{cases}$$

Chapter 24 potentials

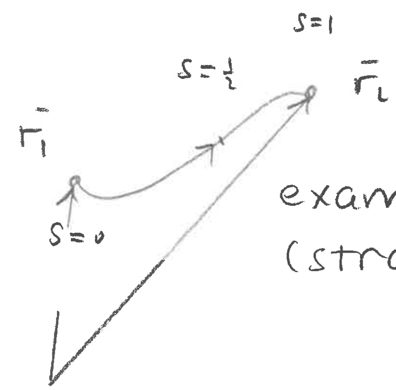
* work done <u>against</u> force increases the potential energy

* consider a charge Q at the origin

consider a path from $\bar{r}_1$ to $\bar{r}_2$

$$\bar{r}(s) = (x(s), y(s), z(s))$$

$$\bar{r}(0) = \bar{r}_1 \qquad \bar{r}(1) = \bar{r}_2$$



example $\bar{r}(s) = \bar{r}_1 + s(\bar{r}_2 - \bar{r}_1)$
(straight line).

$$dV = dW = -\bar{F}\cdot d\bar{r} = -\bar{F}\cdot \frac{d\bar{r}}{ds}\,ds$$

for a charge q

$$= -\left(k\frac{Qq}{r(s)^2}\,\hat{r}(s)\right)\cdot \frac{d\bar{r}}{ds}\,ds$$

$$= -kQq \frac{1}{r^2(s)} \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right) \cdot \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \right) ds$$

$$= -kQq \left(\frac{x \frac{dx}{ds} + y \frac{dy}{ds} + z \frac{dz}{ds}}{(x^2+y^2+z^2)^{3/2}} \right) ds$$

note

$$\frac{d}{ds} (x^2+y^2+z^2)^{-1/2} = -\frac{1}{2} (x^2+y^2+z^2)^{-3/2} \cdot \left(2x \frac{dx}{ds} + 2y \frac{dy}{ds} + 2z \frac{dz}{ds}\right)$$

$$= - \frac{x \frac{dx}{ds} + y \frac{dy}{ds} + z \frac{dz}{ds}}{(x^2+y^2+z^2)^{3/2}}$$

This means

$$\boxed{dW = kQq \frac{d}{ds} (x^2+y^2+z^2)^{-1/2} \cdot ds = du}$$

$$\boxed{dW = kQq \frac{d}{ds} \left(\frac{1}{r} \right) ds}$$

The total work done moving the charge q against the coulomb force is

$$W_{12} = \int_0^1 kQq \frac{d}{ds} \left(\frac{1}{r(s)} \right) ds = kQq \left(\frac{1}{r(1)} - \frac{1}{r(0)} \right)$$

$$= kQq \left(\frac{1}{r_2} - \frac{1}{r_1} \right) = U_2 - U_1$$

observations

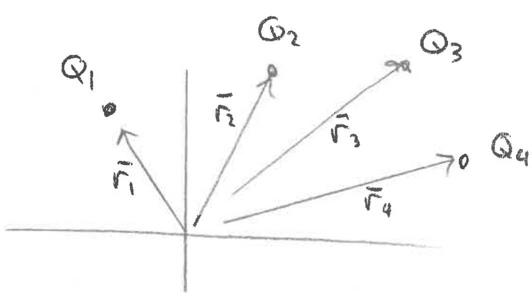
① The work done against the field is independent of path; it only depends on the endpoints

② This means the force is conservative, - the change in potential energy is the work done against the force

$$\boxed{U(\bar{r}_2) - U(\bar{r}_1) = kqQ\left(\frac{1}{\bar{r}_2} - \frac{1}{\bar{r}_1}\right)}$$

③ r_1 and r_2 represent the distace between q and Q at the initial and final points.

Next consider a distribution of charges $Q_1 \cdots Q_N$ at $\bar{r}_1 \cdots \bar{r}_N$



we use superposition

$$dW = -\bar{F}\cdot d\bar{r} = -\sum q\,\bar{E}_i \cdot dr$$

$$\boxed{W_{12} = \sum_{n=1}^{N} k\,q\,Q_n \left(\frac{1}{|\bar{r}_2 - \bar{r}_n|} - \frac{1}{|\bar{r}_1 - \bar{r}_n|} \right) = U_2 - U_1}$$

The charge q is a property of the particle; the rest of the expression is related to the field.

$$\boxed{E = \frac{F}{q} \;\rightarrow\; V_{\substack{\text{electrostatic}\\ \text{potential}}} = \frac{U_{\text{potential energy}}}{q}}$$

* the units of electrostatic potential are

$$\boxed{1 \text{ Volt} = \frac{1 \text{ Joule}}{1 \text{ Coulomb}} = \frac{1 \text{ Newton}\cdot\text{meter}}{1 \text{ coulomb}}}$$

* Just like potential energy — what matters is the <u>difference</u> in electrostatic potential

example point charge q

$$\Delta U = U_2 - U_1 = \frac{kqQ}{r_2} - \frac{kQq}{r_1}$$

$$\Delta V = \frac{U_2}{Q} - \frac{U_1}{Q} = \frac{kq}{r_2} - \frac{kq}{r_1}$$

it is useful to <u>define</u> the potential when $r = \infty$ as 0. Then $U_1 = 0$

$$\boxed{V(r) = \frac{kq}{r} \quad \text{point charge at origin}}$$

$$\boxed{V(r) = \frac{kq}{\sqrt{x^2+y^2+z^2}}}$$

Relation to Electric field

$$\frac{\partial V}{\partial x} = kq\left(-\tfrac{1}{2}\right)(x^2+y^2+z^2)^{-3/2} \cdot (2x)$$

$$= -kq\frac{x}{(x^2+y^2+z^2)^{3/2}} = -kq\frac{x}{r^{3/2}} = -\bar{E}\cdot\hat{x} \doteq -E_x$$

similarly

$$\frac{\partial V}{\partial y} = -kq\frac{y}{r^{3/2}} = -\bar{E}\cdot\hat{y} = -E_y$$

$$\frac{\partial V}{\partial z} = -kq\frac{z}{r^{3/2}} = -\bar{E}\cdot\hat{z} = -E_z$$

$$\bar{E} = \hat{x} E_x + \hat{y} E_y + \hat{z} E_z$$

$$= -\hat{x}\frac{\partial V}{\partial x} - \hat{y}\frac{\partial V}{\partial y} - \hat{z}\frac{\partial V}{\partial z}$$

$$\boxed{\bar{E} = -\bar{\nabla} V}$$ point charge

For many charges we use superposition

$$V(\bar{r}) = \sum_{n=1}^{N} \frac{kQ_n}{|\bar{r}-\bar{r}_N|}$$

$$\bar{E}(r) = \sum_{n=1}^{N} kQ_n \frac{\bar{r}-\bar{r}_N}{|\bar{r}-\bar{r}_N|^3}$$

$$= \sum_{n=1}^{N} kQ_n(-\bar{\nabla})\frac{1}{|r-r_N|}$$

$$= -\bar{\nabla} V$$

(here we used

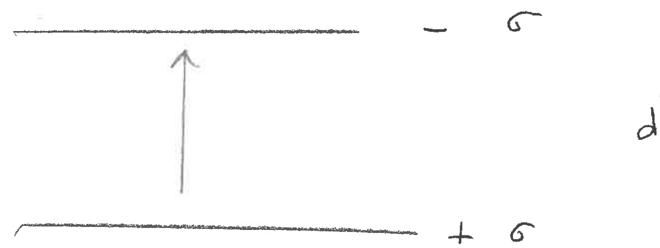
$$\frac{\partial}{\partial x}\frac{1}{\sqrt{(x-x_k)^2+(y-y_n)^2+(z-z_n)^2}} =$$

$$-\frac{1}{2}\left(\sqrt{(x-x_n)^2+(y-y_n)^2+(z-z_n)^2}\right)\left(2(x-x_n)\right)$$

$$= -\frac{x-x_n}{|r-r_k|^{3/2}}$$

etc.

consider a pair of oppositely charged infinite parallel plates with separation d



from the beginning of the lecture the electric field is constant

$$\bar{E} = \frac{\sigma}{\epsilon_0}\hat{z}$$

the work done in moving a charge from the negative plate to the positive plate is

$$U_+ - U_- = W_{-\to+} = q\,\bar{E}\,\hat{z}\cdot(-\hat{z}\,d)$$

$$V_+ - V_- = \frac{W_{-\to+}}{q} = -Ed = -\frac{\sigma d}{\epsilon_0}$$

so the potential difference is

$$V = -\frac{\sigma d}{\epsilon_0}$$

if we only move by z

$$V = -\frac{\sigma z}{\epsilon_0}$$

$$-\nabla V = -\hat{x}\underset{0}{\frac{\partial V}{\partial x}} - \hat{y}\underset{0}{\frac{\partial V}{\partial y}} - \hat{z}\frac{\partial V}{\partial z} = \hat{z}(-1)\left(-\frac{\sigma}{\epsilon_0}\right) = \hat{z}\frac{\sigma}{\epsilon_0}$$

we can use the reverse - if we put a potential difference of 5 volts on the plates then

$$\boxed{\frac{\sigma d}{\epsilon_0} = 5V \qquad \sigma = \frac{\epsilon_0(5V)}{d} \qquad E = \frac{\sigma}{\epsilon_0} = \left(\frac{5V}{d}\right)}$$

electric potential due to a dipole

recall

$$\bar{F} = kq(-\bar{\nabla})\left(\frac{\bar{p}\cdot\bar{r}}{r^3}\right)$$

$$\bar{E} = \frac{\bar{F}}{q} = k(-\bar{\nabla})\left(\frac{\bar{p}\cdot\bar{r}}{r^3}\right)$$

It follows that

$$\boxed{V_{dipole} = k\,\frac{\bar{p}\cdot\bar{r}}{r^3} = \frac{1}{4\pi\epsilon_0}\frac{\bar{p}\cdot\bar{r}}{r^3} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}}$$

potential due to a continuous charge distribution

$$V(r) = k \sum_{n=1}^{N} \frac{q_n}{|\bar{r}-\bar{r}_n|} \rightarrow \qquad \sum q_n \rightarrow \int \rho(r)\,dV$$

$$k \int dV \frac{\rho(\bar{r}')}{|\bar{r}-\bar{r}'|} \qquad \text{(volume charge)}$$

or

$$k \int dA \frac{\sigma(r')}{|\bar{r}-\bar{r}'|} \qquad \text{(surface charge)}$$

$$k \int ds \frac{\lambda(r')}{|r-r'|} \qquad \text{(line charge)}$$

example - circular loop of radius r in xy plane

$$V(r) = k \int_0^{2\pi} \frac{\lambda R\,d\theta}{((x-R\cos\theta)^2+(y-R\sin\theta)^2+z^2)^{1/2}}$$

$$= k \int_0^{2\pi} \frac{\lambda R\,d\theta}{\sqrt{x^2+y^2+z^2+R^2-2xR\cos\theta-2yR\sin\theta}}$$

when $x=y=0$ (on z axis

$$V(0 0 z) = 2\pi k R \lambda \frac{1}{\sqrt{z^2+R^2}}$$

$$E_z = -\frac{\partial}{\partial z} V(0 0 z) = 2\pi k R \lambda \frac{z}{(\sqrt{z^2+R^2})^3}$$