1. A particle of mass $m$ in one dimension experiences a force $F = -kx^3$.
   a. Find the potential of this system.
   b. What is the energy of the system.
   c. If the particle is at rest when $x = d$, what is the maximum speed of the particle?
   d. What is the acceleration of the particle when $x = d$?

   a. The potential is:
   \[ V(x) = -\int_0^x F(x')dx' = -\int_0^x (-kx'^3)dx' = \frac{k}{4}x^4 \]

   b. The energy is:
   \[ E = \frac{k}{4}x^4 + \frac{m}{2}\dot{x}^2 \]

   c. Since the force is conservative energy is conserved. Energy conservation gives
   \[ \frac{k}{4}d^4 = \frac{k}{4}0^4 + \frac{m}{2}x^2 \]
   This can be solved for the maximum speed:
   \[ \dot{x} = \sqrt{\frac{k}{2m}d^2} \]

   d. The acceleration is the force divided by the mass. When $x = d$ the acceleration is
   \[ a = \frac{F}{m} = -\frac{k}{m}d^3 \]

2. Two identical particles of mass $m$ are connected to each other by a spring of force constant $k$. In addition each particle is connected to a wall by another spring of force constant $k$, where the wall connections for each particle are separated by a distance $L$.
   a. Choose a set of generalized coordinates for this system.
   b. Find the equilibrium points.
   c. Check the stability of the equilibrium points.
   d. Find the normal mode frequencies.
   e. Find the normal mode vectors.
a. $x, y =$ distance from each particle to the adjacent wall.

b. To find the equilibrium points the partial derivatives of the potential must all vanish. This requires:

$$V(x, y) = \frac{k}{2}(x^2 + y^2 + (L-x-y)^2) = \frac{k}{2}(2x^2 + 2y^2 + 2x y - 2L(x+6) + L^2)$$

$$\frac{\partial V}{\partial x} = 0 = k(2x + y - L)$$

$$\frac{\partial V}{\partial y} = 0 = k(2y + x - L)$$

These equations are satisfied for $x = y = \frac{L}{3}$

c. The equilibrium is stable since the potential matrix has positive eigenvalues:

$$0 = \det \begin{pmatrix} \lambda - 2k & -k \\ -k & \lambda - 2k \end{pmatrix} \quad (\lambda - 2k) = \pm k \quad \lambda = k, 3k > 0$$

d. The normal mode frequencies are roots of

$$0 = \det \begin{pmatrix} -\omega^2 m + 2k & k \\ k & -\omega^2 m + 2k \end{pmatrix} \quad (\omega^2 m - 2k) = \pm k \quad \omega = \sqrt{\frac{k}{m}}, \sqrt{\frac{3k}{m}} > 0$$

e. Inserting the normal mode frequencies in the matrix above determines equation for the normal mode vectors up to normalization:

$$\begin{pmatrix} -\omega^2 m + 2k & k \\ k & -\omega^2 m + 2k \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

3. Consider a person on the equator of the Earth running due east with speed $v$. Let $R$ be the radius of the Earth and $\omega$ be the Earth’s angular velocity.

a. What is the net force on the person in the Earth fixed coordinate system?

b. What is the direction of the centrifugal force?

c. What is the direction of the Coriolis force?

d. How do these forces affect the gravitational force experienced by the person?
a. Choose coordinates so \( \hat{z} \) is North, \( \hat{x} \) is up and \( \hat{y} \) is East. The force in the radial (up) direction is
\[
F_r = -\frac{GM_e m}{R_e^2} + m\omega^2 R_e + 2m\omega v
\]

b. Radially outward (up):
\[
\mathbf{F} = -m\omega^2 R_e \hat{z} \times (\hat{z} \times \hat{x}) = m\omega^2 R_e \hat{x}
\]

c. Radially outward (up):
\[
\mathbf{F} = -2m\omega v \hat{z} \times \mathbf{y} = 2m\omega v \hat{x}
\]
d. They reduce \( g \) by \( \omega^2 R_e + 2\omega v \)

4. Consider a path between the two points \((0,0)\) and \((a,b)\) in a plane.
   a. Find a functional that gives the length of the path.
   b. Find equations for a path that minimizes the distance.
   c. Show that the solution of these equation is a straight line.
   d. Find the solution.

   a. The length functional is
\[
L = \int ds = \int \sqrt{dx^2 + dy^2} = \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx
\]

b. The Euler Lagrange equations give:
\[
\frac{d}{dx} \left( \frac{d}{dx} \frac{d}{dx} \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right) \right) = 0
\]

c. The first integral is
\[
c = \frac{dy}{dx} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}
\]
\[
\frac{dy}{dx} = \pm \frac{\sqrt{c^2}}{\sqrt{1 - c^2}} = \text{constant}
\]

This means \( \frac{du}{dx} = c' = \text{constant} \), which is the equation of a line with slope \( c' \)

d. The solution satisfying the boundary conditions is
\[
y(x) = \frac{b}{a} x
\]
5. A particle of mass \( m \) orbits the earth in a circular orbit of radius \( r_s \). It interacts with the earth gravitationally.

a. Choose a set of independent generalized coordinates.
b. Find the kinetic energy of the system in terms of your generalized coordinates.
c. Find the Lagrangian of the system
d. Write down Lagrange’s equations for this system
e. Find the period of the orbit as a function of \( R \)

---

a. Since angular momentum is conserved for a central force the motion is in a plane. Choose polar coordinates \((r, \phi)\) in the plane.
b. The kinetic energy is

\[
T = \frac{\mu}{2} (\dot{r}^2 + r^2 \dot{\phi}^2)
\]

where \( \mu = \frac{mM_e}{M_e+m} \) is the reduced mass
c. 

\[
\mathcal{L} = \frac{\mu}{2} (\dot{r}^2 + r^2 \dot{\phi}^2) + \frac{GmM}{r}
\]
d. Lagrange’s equations are

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = 0
\]

\[
\mu \frac{d^2 r}{dt^2} = \mu r \dot{\phi}^2 - \frac{GmM}{r^2}
\]

\[
\frac{d^2 r}{dt^2} = r \dot{\phi}^2 - \frac{G(m + M)}{r^2}
\]
e. For a circular orbit \( \dot{r} = 0 \):

\[
r \dot{\phi}^2 = \frac{G(m + M)}{r^2}
\]

\[
r \left( \frac{2\pi}{T} \right)^2 = \frac{G(m + M_e)}{r^2}
\]

\[
T^2 = \frac{G(m + M_e)}{4\pi^2 r^3}
\]

6. Two particles of mass \( m \) and charge \( q \) are confined to a plane. Choose a set of generalized coordinates.

a. Choose a set of independent generalized coordinates.
b. Find the Lagrangian

c. Find the generalized momenta associated with your choice of generalized coordinates.

d. Find the Hamiltonian

e. Write down Hamiltonians equations for this system

a. Use $X$ and $Y$ coordinates of the center of mass and polar coordinates $r$ and $\phi$ for the relative coordinates. Note $M = 2m$ and $\mu = m/2$.

b. 
\[ L = T - V = \frac{1}{2}(2m)(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{q^2}{4\pi \epsilon_0 r} \]

c. 
\[ P_X = 2m\dot{X}, \quad P_Y = 2m\dot{Y}, \quad p_r = \frac{m\dot{r}}{2}, \quad p_\phi = \frac{mr^2\dot{\phi}}{2} \]

d. 
\[ H = \frac{P_X^2}{4m} + \frac{P_Y^2}{4m} + \frac{p_r^2}{mr^2} + \frac{q^2}{4\pi \epsilon_0 r^2} \]

e. 
\[ \dot{P}_X = -\frac{\partial H}{\partial X} = 0 \quad \dot{X} = \frac{\partial H}{\partial P_X} = \frac{P_X}{2m} \]
\[ \dot{P}_Y = -\frac{\partial H}{\partial Y} = 0 \quad \dot{Y} = \frac{\partial H}{\partial P_Y} = \frac{P_Y}{2m} \]
\[ \dot{P}_\phi = -\frac{\partial H}{\partial \phi} = 0 \quad \dot{\phi} = \frac{\partial H}{\partial p_\phi} = 2 \frac{p_\phi}{mr^2} \]
\[ \dot{p}_r = -\frac{\partial H}{\partial r} = 2 \frac{p_\phi^2}{mr^3} + \frac{q^2}{4\pi \epsilon_0 r^2} \quad \dot{r} = \frac{\partial H}{\partial p_r} = \frac{2p_r}{m} \]