3 dimensional isotropic harmonic oscillator

\[ V = \frac{1}{2} k \mathbf{\vec{r}} \cdot \mathbf{\vec{r}} \]

\[ \mathbf{\vec{F}} = -\nabla V = -k \mathbf{\vec{r}} = -k (\mathbf{\hat{i}}x + \mathbf{\hat{j}}y + \mathbf{\hat{k}}z) \]

The second law has three components

\[ m \frac{d^2 \mathbf{\vec{r}}}{dt^2} = \mathbf{\vec{F}} \]

\[ m \frac{d^2 x}{dt^2} = -kx \]

\[ m \frac{d^2 y}{dt^2} = -ky \]

\[ m \frac{d^2 z}{dt^2} = -kz \]

This looks like three one-dimensional harmonic oscillators with the same angular velocity

\[ \omega^2 = \frac{k}{m} \]

The solutions have the form

\[ x(t) = C_x \cos(\omega t) + d_x \sin(\omega t) \]

\[ y(t) = C_y \cos(\omega t) + d_y \sin(\omega t) \]

\[ z(t) = C_z \cos(\omega t) + d_z \sin(\omega t) \]

We define

\[ \mathbf{\vec{\mathbf{C}}} = C_x \mathbf{\hat{i}} + C_y \mathbf{\hat{j}} + C_z \mathbf{\hat{k}} \]
\[ \ddot{d} = d_x \hat{i} + d_y \hat{j} + d_z \hat{k} \]

With this notation the solution has the form

\[ \ddot{r}(t) = \ddot{c} \cos(\omega t) + \ddot{d} \sin(\omega t) \]

The six constants \( \ddot{c} \) and \( \ddot{d} \) are fixed by the initial coordinate and velocity of each oscillator.

By momentum conservation

\[ \ddot{p}(t) = m \ddot{r} = \]

\[ -m \ddot{c} \omega \sin(\omega t) + m \ddot{d} \omega \cos(\omega t) \]

\[ \dot{J} = \ddot{r}(t) \times \ddot{p}(t) = \]

\[ (\ddot{c} \cos(\omega t) + \ddot{d} \sin(\omega t)) \times \]

\[ (-m \ddot{c} \omega \sin(\omega t) + m \ddot{d} \omega \cos(\omega t)) = \]

\[ = m \omega \ddot{c} \dot{x} \ddot{d} \cos^2(\omega t) + \]

\[ -m \omega \ddot{d} \dot{x} \ddot{c} \sin^2(\omega t) + \]

\[ (-m \omega \ddot{c} \dot{x} \ddot{c} + m \omega \ddot{d} \dot{x} \ddot{d}) \sin(\omega t) \cos(\omega t) \]

\[ \dot{J} = m \omega \ddot{c} \dot{x} \ddot{d} \]

We see that \( \dot{J} \) is independent of time and is \( \perp \) to the plane containing \( \ddot{r}(t) \) and \( \ddot{p}(t) \).
Energy conservation

\[ E = \frac{1}{2} m \dot{\vec{r}}^2 + \frac{1}{2} k \vec{r}^2 = \]

\[ \frac{1}{2} m \left( -\ddot{\vec{c}} \omega \sin(\omega t) + \ddot{\vec{d}} \omega \cos(\omega t) \right) + \]

\[ \frac{1}{2} k \left( \dot{\vec{c}} \cos(\omega t) + \dot{\vec{d}} \sin(\omega t) \right) \]

\[ \left( \frac{1}{2} m \vec{c}^2 \omega^2 + \frac{1}{2} k \vec{d}^2 \right) \sin^2(\omega t) + \]

\[ \left( \frac{1}{2} m \omega^2 + \frac{1}{2} k \vec{c}^2 \right) \cos^2(\omega t) \]

\[ \left( \frac{1}{2} m \omega^2 \left( -2 \ddot{\vec{c}} \cdot \vec{d} + \frac{1}{2} k (\vec{c} \cdot \vec{d}) \right) \sin(\omega t) \cos(\omega t) \right) \]

Note

\[ \frac{1}{2} m \omega^2 = \frac{1}{2} m \frac{k}{m} = \frac{1}{2} k \]

Using this in the above gives

\[ \frac{1}{2} k \left( \vec{c}^2 + \vec{d}^2 \right) \left( \sin^2 \omega t + \cos^2 \omega t \right) \]

\[ \frac{1}{2} k \left( 2 \vec{c} \cdot \vec{d} - 2 \vec{c} \cdot \vec{d} \right) \sin \omega t \cos \omega t = \]

\[ \frac{1}{2} k \left( \vec{c}^2 + \vec{d}^2 \right) \]

\[ E = \frac{1}{2} k \left( \vec{c}^2 + \vec{d}^2 \right) = \text{constant} \]
This shows that the motion is in a plane with energy and angular momentum conserved:

\[ \ddot{r} = m \omega \bar{e} \times \bar{a} \]
\[ E = \frac{1}{2} k (\bar{e}^2 + \bar{a}^2) \]

To find properties of the orbit consider an alternative set of independent solutions:

\[ \bar{r}(t) = \bar{a} \cos(\omega t - \phi) + \bar{b} \sin(\omega t - \phi) \]

where \( \phi \) is an arbitrary phase (no matter how we choose \( \phi \) the solutions are independent).

To find the relation between \( \bar{a}, \bar{b}, \bar{e}, \bar{a} \) use:

\[ \cos(A - B) = \cos(A) \cos(B) + \sin(A) \sin(B) \]
\[ \sin(A - B) = \sin(A) \cos(B) - \sin(B) \cos(A) \]

\[ \bar{r}(t) = \bar{a} \cos(\omega t) \cos \phi + \bar{a} \sin(\omega t) \sin \phi \]
\[ + \bar{b} \sin(\omega t) \cos \phi - \bar{b} \cos(\omega t) \sin \phi \]

\[ = (\bar{a} \cos(\phi) - \bar{b} \sin(\phi)) \cos(\omega t) \]
\[ + (\bar{a} \sin(\phi) + \bar{b} \cos(\phi)) \sin(\omega t) \]
\[ \begin{align*}
\dot{c} &= \ddot{a} \cos(\Phi) - \ddot{b} \sin(\Phi) \\
\ddot{d} &= \ddot{a} \sin(\Phi) + \ddot{b} \cos(\Phi)
\end{align*} \]

to invert this multiply the first equation by \( \cos \Phi \) and the second equation by \( \sin \Phi \) and add

\[ \dot{c} \cos \Phi + \ddot{d} \sin \Phi = \ddot{a} \cos^2 \Phi + \ddot{b} \sin^2 \Phi + \dddot{a} \]

\[ \dddot{a} = \dot{c} \cos \Phi + \ddot{d} \sin \Phi \]

multiply the first equation by \( -\sin \Phi \) and the second by \( \cos \Phi \) and add

\[ -\dot{c} \sin \Phi + \ddot{d} \cos \Phi = \ddot{b} (\sin^2 \Phi + \cos^2 \Phi) + \dddot{a} \]

\[ \dddot{b} = \ddot{d} \cos \Phi - \dot{c} \sin \Phi \]

the initial data fixed \( \dddot{c} \) and \( \dddot{d} \), we are free to choose the phase \( \Phi \) choose it to make \( \dddot{a} = \dddot{b} \)

\[ \Phi = \dot{a} \cdot \dot{b} = (\dot{c} \cos \Phi + \dot{d} \sin \Phi) \cdot (\ddot{d} \cos \Phi - \ddot{c} \sin \Phi) \]

\[ = \dot{c} \ddot{d} \left( \cos^2 \Phi - \sin^2 \Phi \right) + \sin \Phi \cos \Phi \left( \dddot{d}^2 - \dddot{c} \right) \]

\[ \cos 2\Phi \quad \frac{1}{2} \sin 2\Phi \]

\[ \dot{c} \cdot \ddot{d} \cos (2\Phi) = \frac{1}{2} \sin (2\Phi) \left( \dddot{c}^2 - \dddot{d}^2 \right) \]

\[ \tan (2\Phi) = \frac{2 \dot{c} \cdot \ddot{d}}{\dddot{c}^2 - \dddot{d}^2} \]
we are still free to choose a coordinate system

Note

\[ \mathbf{\bar{E}} = \mathbf{m} \omega \mathbf{\bar{c}} \times \mathbf{\bar{d}} = \mathbf{m} \omega \mathbf{\bar{a}} \times \mathbf{\bar{b}} \]

\[ \mathbf{\bar{E}} = \frac{1}{2} k ( \mathbf{\bar{c}}^2 + \mathbf{\bar{d}}^2 ) = \frac{1}{2} k ( \mathbf{\bar{a}}^2 + \mathbf{\bar{b}}^2 ) \]

since \( \mathbf{\bar{a}} + \mathbf{\bar{b}} \) we choose the x axis parallel to \( \mathbf{\bar{a}} \) and the y axis parallel to \( \mathbf{\bar{b}} \). this puts the angular momentum parallel to the \( \mathbf{\bar{z}} \) axis.

\[ x = a \cos (\omega t - \phi) \]
\[ y = b \sin (\omega t - \phi) \]

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 (\omega t) + \sin^2 (\omega t) = 1 \]

This shows that the orbit is a parabola, with major and minor axes \( \mathbf{\bar{a}} \) and \( \mathbf{\bar{b}} \).

we can express \( \mathbf{\bar{a}} \mathbf{\bar{b}} \) in terms of \( \mathbf{\bar{c}} \mathbf{\bar{d}} \)

\[ \mathbf{m} \omega \mathbf{\bar{c}} \times \mathbf{\bar{d}} = \mathbf{m} \omega \mathbf{\bar{a}} \times \mathbf{\bar{b}} \mathbf{\hat{k}} \]

\[ \frac{1}{2} k ( \mathbf{\bar{a}}^2 + \mathbf{\bar{b}}^2 ) = \frac{1}{2} k ( \mathbf{\bar{c}}^2 + \mathbf{\bar{d}}^2 ) \]

\[ \mathbf{\bar{a}} \mathbf{\bar{b}} = \mathbf{\hat{k}} \cdot ( \mathbf{\bar{c}} \times \mathbf{\bar{d}} ) \]

\[ a^2 + b^2 = (\mathbf{\bar{c}}^2 + \mathbf{\bar{d}}^2) \]
\[ b = \frac{1C \times d^1}{a} \]
\[ a^2 + \frac{1C \times d^1}{a^2} = c^2 + d^2 \]

multiply by \(a^2\)
\[ a^4 - a^2(c^2 + d^2) + 1C \times d^1 = 0 \]
\[ a^2 = \frac{(c^2 + d^2) \pm \sqrt{(c^2 + d^2) - 41C \times d^1}}{2} \]

since these equations are symmetric under interchanging \(a, b\) -
one of the roots is \(a^2\) and
one of the roots is \(b^2\)

**General case** \(V(\vec{F}) = V(r)\)

since \(\vec{F} = -\nabla V = -\frac{\partial V}{\partial r} \vec{r} = -\frac{\partial V}{\partial r} \nabla \sqrt{x^2 + y^2 + z^2} \]
\[ = -\frac{\partial V}{\partial r} \vec{r} \]
\[ \vec{F} \times \vec{F} = -\frac{\partial V}{\partial r} \vec{F} \times \vec{r} = 0 \]

so angular momentum is conserved, the motion is necessarily in a plane \(\perp\) to \(\vec{F}\)
we use \( r, \theta \) as coordinates in the plane

\[
T = \frac{1}{2} m ( \dot{r}^2 + r^2 \dot{\theta}^2 ) \\
V = V(r)
\]

Energy conservation gives

\[
E = T + V = \frac{1}{2} m ( \dot{r}^2 + r^2 \dot{\theta}^2 ) + V(r)
\]

The angular momentum

\[
\vec{J} = m \vec{r} \times \dot{\vec{r}}
\]

The component of \( \dot{\vec{r}} \) to \( \vec{F} \) is \( r \dot{\theta} \)

so the magnitude of \( \vec{J} \) is

\[
m r^2 \dot{\theta} = J
\]

We can use the angular momentum conservation to eliminate \( \dot{\theta} \)

\[
E = \frac{1}{2} m ( \dot{r}^2 + r^2 \left( \frac{1}{m r^2} \right)^2 ) + V(r)
\]

\[
E = \frac{1}{2} m \dot{r}^2 + V(r) + \frac{J^2}{2 m r^2}
\]
This looks like a one dimensional motion in a potential

$$V_{\text{eff}}(r) = V(r) + \frac{\mathbf{J}^2}{2m} \frac{1}{r^2}$$

This is called the effective potential.

We can always reduce this to a first order equation in $r$

$$r^2 = \frac{2}{m} \left( E - V(r) - \frac{\mathbf{J}^2}{2m} \frac{1}{r^2} \right)$$

$$\frac{dr}{dt} = \pm \sqrt{r \left( E - V(r) - \frac{\mathbf{J}^2}{2m} \frac{1}{r^2} \right)}$$

$$dt = \pm \frac{dr}{\sqrt{r \left( E - V(r) - \frac{\mathbf{J}^2}{2m} \frac{1}{r^2} \right)}}$$

$$t - t_0 = \pm \int_{r_0}^{r} \frac{dr'}{\sqrt{r' \left( E - V(r') - \frac{\mathbf{J}^2}{2m} \frac{1}{r'} \right)}}$$

We still have to be able to do the integral and solve for $r(t)$. 

we can also calculate the orbit

\[
\dot{\theta} = \frac{d\theta}{dt} = \frac{d\theta}{dr} \frac{dr}{dt} = \frac{d\theta}{dr} \cdot r'
\]

\[
\frac{d\theta}{dr} = \frac{\dot{r}}{r} = \frac{J}{m^2} \sqrt{\frac{1}{2} \left( \frac{1}{\sqrt{E-V(r)}} - \frac{x^2}{2m^2 r^2} \right)}
\]

\[
\int d\theta = \frac{J}{\sqrt{2m}} \int \frac{dr}{r^2} \frac{1}{\sqrt{E-V(r)} - \frac{x^2}{2m^2 r^2}}
\]

\[
\Theta(r) - \Theta(r_e) = \frac{J}{\sqrt{2m}} \int_{r_e}^{r} \frac{dr}{r^2} \frac{1}{\sqrt{E-V(r)} - \frac{x^2}{2m^2 r^2}}
\]

The effective potential can be used like an ordinary potential

\[
\frac{\partial V_{\text{eff}}}{\partial r}(r_e) = 0 \quad r_e
\]

corresponds to an equilibrium point

\[
\frac{\partial^2 V_{\text{eff}}}{\partial r^2}(r_e) > 0
\]

means \( r_e \) is a stable equilibrium which means that we will have small oscillations about \( r_e \). The motion will be that of an isotropic harmonic oscillator with spring constant \( \frac{\partial^2 V}{\partial r^2}(r_e) = \cdot \cdot \cdot \)
An important case in physics is inverse square forces. This is because both the gravitational and coulomb forces are inverse square forces:

\[ V_{\text{grav}}(r) = -\frac{GM_1M_2}{r} \]

which gives an attractive force

\[ F(r) = -\frac{GM_1M_2}{r^3} \]

and the coulomb force

\[ V_c(r) = \frac{q_1q_2}{4\pi\varepsilon_0 r} \]
\[ F_c(r) = \frac{q_1q_2}{4\pi\varepsilon_0 r^2} \]

which is repulsive if \( q_1q_2 > 0 \) and attractive if \( q_1q_2 < 0 \).

We treat both cases by assuming

\[ V(r) = \frac{kr}{r} \]

\( k > 0 \) repulsive \( k < 0 \) attractive.

Energy and angular momentum conservation give

\[ mr^2 \dot{\theta} = \mathcal{J} \]
\[ \frac{1}{2} m\dot{r}^2 + \frac{k}{r} + \frac{\mathcal{J}^2}{2mr^2} = E \]
The one difference from the one-dimensional case is that $r \geq 0$ since it represents the distance from the origin.

If $r > 0$

(i) the effective potential is always positive

(ii) it is infinite at $r = 0$ and falls off monotonically as $r$ increases.

It is clear in this case there are no stable equilibrium points - the slope is always decreasing.

If a particle approaches the origin from $r = \infty$ at that point $V = 0$ and

\[ E = \frac{1}{2} m \dot{r}^2 \]

\[ T = m r^2 \dot{\theta} \]
If we calculate $J$ using $mV \times \hat{r}$ component of $F$:

$$J = mVb$$

when the particle is very away from the center, this has to be:

$$mVb = mr^2 \hat{r} - J$$

since the left side is finite the right side is finite.

We also have energy conservation:

$$E = \frac{1}{2} mV^2 = \frac{1}{2} m \dot{r}^2 + \frac{k}{r} + \frac{m^2 v^2 b^2}{2m} \frac{1}{r^2}$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{k}{r} + \frac{m^2 v^2 b^2}{2m} \frac{1}{r^2}$$

since the potential is infinite, positive at the origin - eventually $r$ must change sign when this happens.

$$\frac{1}{2} mV^2 = 0 + \frac{k}{r} + \frac{m^2 v^2 b^2}{2} \frac{1}{r^2}$$

multiply by $r^2 \frac{2}{mV^2}$

$$r^2 - \frac{2k}{mV^2} r - b^2 = 0$$

$$r^2 - \frac{k}{E} r - b^2 = 0$$
This has 2 roots:

\[ r_{\text{min}} = \frac{1}{2} \left( \frac{k}{E} \pm \sqrt{\frac{k^2}{E^2} + 4b^2} \right) \]

Since \( \sqrt{\frac{k^2}{E^2} + 4b^2} > \left( \frac{k}{E} \right) \)

Only one root is positive.

The distance of closest approach is

\[ r_{\text{min}} = \frac{1}{2} \frac{k}{E} \left( 1 + \sqrt{1 + \frac{4bE^2}{k^2}} \right) \]

When \( k < 0 \) the potential is attractive.

\[ V_{\text{eff}} = -\frac{k}{r} + \frac{J^2}{2mr^2} \]

In this case we look for an equilibrium point:

\[ \frac{\partial V}{\partial r} = \frac{k}{r^2} - 2 \frac{J^2}{2mr^3} = 0 \]

Solving:

\[ r(R) = \frac{J^2}{mkr} \]

\[ \frac{\partial^2 V}{\partial r^2} = -2 \frac{k}{r^3} + 6 \frac{J^2}{2mr^4} \]

\[ r_{\text{eq}} = \frac{J^2}{mk^2} \]

\[ = -2 \frac{k}{J^2} m + 6 \frac{J^2}{2m} \frac{m^3}{k^2} \]

\[ = \frac{k^4 m^3}{J^6} > 0 \]
The effective potential looks like

\[ -\frac{\mu}{2r^2} \]

The potential energy at that value of \( r \) is

\[ V = -\frac{\mu}{r} - \frac{J^2}{2mr^2} \]
\[ = -\frac{\mu}{r} - \frac{J^2}{2m} \frac{\mu^2}{r^4} \]
\[ = -\frac{\mu}{2} \frac{\mu^3}{r^2} \]

In energies between \( -\frac{1}{2} \frac{\mu^3}{r^2} \) and 0 there will be orbital motion between the extrem points where

\[ E = \frac{1}{2} m \dot{r}^2 - \frac{\mu}{r} + \frac{J^2}{2mr^2} \]

Setting \( \dot{r} = 0 \)

\[ 2mE + 2mJ^2 - J^2 = 0 \]
\[ r^2 + \frac{\mu}{E} r - \frac{J^2}{2mE} = 0 \]
\[ r_+ = \frac{|k|}{2E} \pm \sqrt{\frac{|k|^2}{4E^2} + \frac{J^2}{2mE}} \]

Note here \( E \) is negative

\[ r_+ = \frac{|k|}{2E} + \sqrt{\frac{|k|^2}{4E^2} - \frac{J^2}{2mE}} \]

\[ r_- = \frac{|k|}{2E} - \sqrt{\frac{|k|^2}{4E^2} - \frac{J^2}{2mE}} \]

If

\[ \frac{|k|^2}{4E^2} = \frac{J^2}{2mE} \]

\[ E = \frac{|k|m}{2} \frac{J^2}{|k|^2} \]

Then \( r_+ = r_- \) in this case there is a single circular orbit at

\[ r = \frac{|k|}{2E} \]