In this problem the conserved angular momentum is
\[ J = Rm v \sin \alpha \]

The conserved energy is
\[ E = \frac{1}{2} m v^2 - \frac{|k|}{R} \quad |k| = GM_\text{e}m \]

The conservation laws give
\[ m r^2 \dot{\theta} = J \]
\[ \frac{1}{2} m (r^2 + r^2 \dot{\theta}^2) - \frac{|k|}{R} = E \]

The maximum distance from the center of the earth will be finite if \( E < 0 \) - the means
\[ \frac{|k|}{R} > \frac{1}{2} m v^2 \]. It can be determined by setting \( \dot{r} = 0 \) in the energy conservation equation
\[ \frac{1}{2} m r^2 \left( \frac{J^2}{m^2 r^4} \right) - \frac{|k|}{R} = \frac{1}{2} m v^2 - \frac{|k|}{R} \]
solving for \( r \)

\[
0 = \left( \frac{1}{2} m v^2 - \frac{|k|}{R} \right) + \frac{|k|}{r} - \frac{J^2}{2 m r^2}
\]

or

\[
\left( \frac{1}{2} m v^2 - \frac{|k|}{R} \right) r^2 + |k| r - \frac{J^2}{2m} = 0
\]

the quadratic equation has 2 roots

\[
r = -\frac{1}{2 E_1} \left( -|k| \pm \sqrt{|k|^2 - 4 \left( -\frac{J^2}{2 E_1} \right)} \right)
\]

\[
= \frac{|k|}{2 E_1} \pm \sqrt{\left( \frac{|k|}{2 E_1} \right)^2 - \frac{J^2}{2m E_1}}
\]

this has 2 positive roots - this makes sense because the orbit is elliptical.

The longer one corresponds to the desired height.
shape of orbits

Start with

\[ \dot{e} = \frac{J}{m r^2} \]

\[ \frac{1}{2} m (r^2 + r^2 \left( \frac{J^2}{m^2 r^4} \right)) + \frac{k}{r} = E \]

The orbit can be expressed as \( r(\theta) \). To find this let

\[ u = \frac{1}{r} \quad \frac{du}{dr} = -\frac{1}{r^2} \quad \frac{dr}{du} = -r^2 \]

\[ \frac{dr}{dt} = \frac{dr}{du} \frac{du}{d\theta} \frac{d\theta}{dt} = (-r^2) \frac{du}{d\theta} \left( \frac{J}{m r^2} \right) \]

\[ = -\frac{J}{m} \frac{du}{d\theta} \]

Using this in the energy equation gives

\[ \frac{1}{2} m \left( \frac{J^2}{m^2} \left( \frac{du}{d\theta} \right)^2 \right) + \frac{1}{2 m} \frac{1}{r^2} + \frac{k}{r} = E \]

\[ \frac{J^2}{2m} \left( \left( \frac{du}{d\theta} \right)^2 + u^2 \right) + k u = E \]

Multiply by \( \frac{2m}{J^2} \)

\[ \left( \frac{du}{d\theta} \right)^2 + u^2 + 2 \frac{k m}{J^2} u = \frac{2m E}{J^2} \]
since \( u \) has dimension \( \frac{1}{u^2} \) and \( \frac{k}{J^2} \) \( u \) have the same dimension \( \Rightarrow \)

\[ l = \frac{J^2}{1k\text{m}} = \text{is a constant with dimension length - using } 1k\text{m} \]

makes this positive independent of the sign of \( k \)

\[ l \equiv \frac{J^2}{1k\text{m}} \]

using this in the equation for \( u(0) \)

\[ \left( \frac{du}{d\theta} \right)^2 + u^2 \pm \frac{2}{\rho} u = \left( \frac{1k\text{m}}{J^2} \right) \left( 2 \frac{E}{1\text{m}} \right) = \frac{1}{\rho} 2 \frac{E}{1\text{m}} \]

here the upper sign (\( + \)) is

for \( k > 0 \) the lower sign (\( - \)) is

for \( k < 0 \).

Define the dimensionless quantity

\[ z = lu \]
multiply the equation by \( e^2 \)
using \( \dot{u} = Z \ e^u = \dot{Z} \) gives

\[
\left( \frac{dZ}{d\theta} \right)^2 + Z^2 \pm 2Z = \frac{2E^2}{1|\theta|}
\]

\[
\left( \frac{dZ}{d\theta} \right)^2 + (Z \pm 1)^2 - 1 = \frac{2E^2}{1|\theta|}
\]

\[
\left( \frac{dZ}{d\theta} \right)^2 + (Z \pm 1)^2 = \left( 1 + \frac{2E^2}{1|\theta|} \right)
\]

since the left side of this equation is a sum of squares it is positive - this means that for allowed values of \( E \), the right side should be positive

\[
e^2 = \left( 1 + \frac{2E^2}{1|\theta|} \right)
\]

from this expression

\( E > 0 \rightarrow e^2 > 1 \)

\( E < 0 \rightarrow e^2 < 1 \)

\( E = 0 \rightarrow e^2 = 1 \)
using this in the equation

\[ \left( \frac{dZ}{du} \right)^2 + (Z \pm 1)^2 = e^2 \]

This is a simple equation to solve. The general solution is

\[ Z \pm 1 = e \cos (\theta - \theta_0) \]

It is easy to check that this \( z(\theta) \) satisfies the differential equation. Using \( Z = x u = \frac{e}{r} \) gives an equation for \( r \)

\[ \frac{e}{r} = e \cos (\theta - \theta_0) \pm 1 \]

Remarks

1. If \( e = 0 \) we must use the lower sign (attractive force) which require \( \theta = \pi \)

This corresponds to a circular orbit of radius \( r = e \).
(2) In the attractive case (lower sign) with \( E < 0 \) \((e < 1)\)

\[
\frac{\ell}{r} = 1 + e \cos (\theta - \theta_0)
\]

\[
\frac{\ell}{r} = \frac{\ell}{1 + e \cos (\theta - \theta_0)}
\]

\( r_{\text{max}} = \frac{\ell}{1 - e} \quad r_{\text{min}} = \frac{\ell}{1 + e} \)

\( r_{\text{min}} \) corresponds to \( \theta_0 = \theta_0 \)

\( r_{\text{max}} \) corresponds to a closed orbit.

(3) For \( e > 1 \) (either sign)

\[
\frac{\ell}{r} = e \cos (\theta - \theta_0) \pm 1
\]

the angles where \( \frac{\ell}{r} > 0 \)

are limited to

\[
\cos (\theta - \theta_0) > \pm \frac{1}{e}
\]
the equation
\[ \frac{l}{r} = e \cos(\theta - \omega) + 1 \]
is an equation for a conic section
\[ \frac{l}{r} = e \cos(\theta - \omega) + 1 \quad e < 1 \quad \text{ellipse} \]
\[ \frac{l}{r} = e \cos(\theta - \omega) + 1 \quad e > 1 \quad \text{hyperbola} \]
\[ \frac{l}{r} = \cos(\theta - \omega) + 1 \quad e = 1 \quad \text{parabola} \]
To put these in a more familiar form multiply by \( r \)
\[ l = r e \cos(\theta - \omega) + r \]
choose coordinates so \( \omega = 0 \rightarrow \) (this puts \( r_{\text{min}} \) on the x-axis)
\[ l = r e \cos \theta + r \]
\[ (l - r e \cos \theta) = r \]
let \( x = r \cos \theta \)
\[ (l - ex) = r \]
squaring
\[ l^2 - 2exl + e^2 x^2 = r^2 = x^2 + y^2 \]
we write this as

\[ l^2 = y^2 + (1-e^2)x^2 + 2ex \]

complete the square in \( x \)

\[ l^2 = y^2 + (1-e^2)(x^2 + 2 \frac{e}{1-e^2} x) \]
\[ = y^2 + (1-e^2)(x + \frac{e}{1-e^2})^2 - \frac{e^2}{1-e^2} \]

\[ l^2(1+\frac{e^2}{1-e^2}) = \frac{l^2}{1-e^2} = y^2 + (1-e^2)(x + \frac{e}{1-e^2})^2 \]

multiply by \( \frac{1-e^2}{e^2} \)

\[ 1 = \left(\frac{1-e^2}{e^2}\right)y^2 + \left(\frac{1-e^2}{e^2}\right)(x - \frac{e}{1-e^2})^2 \]

the signs depend on \( e = \frac{e^2-1}{e} \)

If \( e^2 = 1 \) (the equation on the top of the page gives

\[ l^2 = y^2 + 2lx \]

\[ y^2 = l(e-2x) \]

parabolic case
If \( e^2 > 1 \), \( 1 - e^2 \) is negative
\[ e^2 - 1 > 0 \]

\[
1 = -\frac{e^2-1}{b^2} \ y^2 + \left(\frac{e^2-1}{e^2}\right)^2 \left( x + \frac{e\ell}{e^2-1} \right)^2
\]
\[
-\frac{1}{b^2} \ \frac{1}{a^2} \ ea
\]

\[
1 = \frac{1}{a^2} (x+ea)^2 - \frac{y^2}{b^2}
\]

which is the equation for a hyperbola.

If \( e^2 < 1 \)

\[
1 = \frac{1-e^2}{b^2} \ y^2 + \left(1-e^2\right)^2 \left( x - \frac{e\ell}{1-e^2} \right)^2
\]
\[
\frac{1}{b^2} \ \frac{1}{a^2} \ (x - ea)^2
\]

\[
1 = \frac{y^2}{b^2} + \frac{(x-ea)^2}{a^2}
\]

which is the equation for an ellipse with center at \((0,0,0)\).
The parameters $a$ and $b$ are

$$a = \frac{I}{11-e^2} \quad b = l a$$

$$l = \frac{J^2}{11e_l m} \quad 1-e^2 = -\frac{2e_l}{11l}$$

$$a = \frac{11l}{2e_l} \quad b^2 = \frac{11e_l^2}{2e_l} = \frac{J^2}{2m e_l}$$

Orbital motion

From Kepler's 2nd law

$$\frac{dA}{dt} = \frac{I}{2m}$$

which can be integrated to get

$$T = \frac{2m}{J} A$$

where $A$ is the area of the ellipse

$$A = 2 \int_{-a}^{a} \sqrt{1 - \frac{x^2}{a^2}} \, dx$$

$$= 2ab \int_{-a}^{a} \sqrt{1 - \left(\frac{x}{a}\right)^2} \, dx$$

Let $u = x/a$

$$= 2ab \int_{-1}^{1} \sqrt{1 - u^2} \, du$$

$\pi/2 = \text{area of unit semi circle}$
mis is

$$A = \pi ab = \pi a \sqrt{ae}$$

$$A^2 = \pi^2 a^3 e$$

$$T^2 = \frac{4m^2}{J^2} A^2 = \frac{4m^2}{J^2} \pi^2 a^3 e$$

$$T^2 = 4\pi^2 \frac{m}{J^2} e^3$$

(note a only depends on the energy - not $J^2$) This is Kepler's 3rd law - in a circular orbit $a$ is the radius - this is special to inverse square forces.

Hyperbolic motion and scattering

$$\frac{e}{r} = e \cos \theta \neq 1 \quad e > 1 \quad \{ - \frac{k}{e}, \quad + \frac{k}{e}$$

as $r \to \infty$, the right side must vanish from the positive side ($r > 0$)
This requires
\[ \cos(\theta) = \pm \frac{1}{\epsilon} \]

\[ \cos^{-1}(-\frac{1}{\epsilon}) = \pi - \Theta \quad \Theta = \cos^{-1}(\frac{1}{\epsilon}) \]

Since \( \epsilon > 1 \), the physical region is
\[ \Theta \leq \cos^{-1}(\frac{1}{\epsilon}) \]

There are 2 orbits

The scattering angle measures the angular displacement relative to the incoming direction

\[ \phi = \pi - 2\Theta \quad \Theta = \frac{\pi - \phi}{2} \]

(This is true in both cases)
mis relates the scattering angle to $e$

$$\pm \frac{1}{e} = \cos\left(\frac{\pi - \phi}{2}\right)$$

recall $\cos(A - B) = \cos A \cos B + \sin A \sin B$

so $A = \frac{\pi}{2}$ and $B = \frac{\phi}{2}$ mis becomes

$$\sin\left(\frac{\phi}{2}\right)$$

$$\sin\left(\frac{\phi}{2}\right) = \pm \frac{1}{e}$$

$$e^2 = \frac{1}{\sin^2\left(\frac{\phi}{2}\right)}$$

$$e^2 - 1 = \frac{1}{\sin^2\left(\frac{\phi}{2}\right)} - 1 = \frac{1 - \sin^2\left(\frac{\phi}{2}\right)}{\sin^2\left(\frac{\phi}{2}\right)} = \left(\frac{\cos\left(\frac{\phi}{2}\right)}{\sin\left(\frac{\phi}{2}\right)}\right)^2$$

using

$$(e^2 - 1) = \frac{e}{a} = \frac{ea}{a^2} = \frac{b^2}{a^2}$$

gives

$$b^2 = a^2 \cot^2\left(\frac{\phi}{2}\right)$$

$$a = \frac{1k_1}{2} \cdot \frac{1}{2mv^2} = \frac{1k_1}{mv^2}$$

note

$$b^2 = \frac{x^2}{2m^2} = \frac{(bmv_1)^2}{2m(\frac{1}{2}mv^2)} = b^2$$

we see that $b$ is the impact parameter
scattered from an inverse square potential

\[ b^2 = \frac{1}{m^2 v^4} \cot^2 \left( \frac{\phi}{2} \right) \]

\[ b = \text{impact parameter} \]

\[ \phi = \text{scattering angle} \]

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cross sections - sphere of radius \( R \)

the sectional area of a sphere of radius \( R \) is \( \sigma = \pi R^2 \)

consider a perfectly elastic collision \( b = \text{impact parameter}, \)

\( \theta = \text{scattering angle} \)

\[ b = R \sin \alpha \]

\[ \theta = \pi - 2 \alpha \]

\( \alpha = \left( \frac{\pi - \theta}{2} \right) \)

use \( \sin (A-B) = \sin A \cos B - \sin B \cos A \)
This gives
\[ \sin \left( \frac{\pi - \omega}{2} \right) = \cos \left( \frac{\theta}{2} \right) \]

\[
\begin{align*}
\sigma = R \cos \left( \frac{\theta}{2} \right) \quad \text{hard sphere}
\end{align*}
\]

\[ d\sigma = b \, db \, d\phi \]

This is called the differential cross section

\[
\begin{align*}
b &= R \cos \left( \frac{\theta}{2} \right) \\
d\rho &= -R \sin \left( \frac{\theta}{2} \right) \frac{1}{2} \, d\phi \\
bd\phi d\phi &= -\frac{R^2}{2} \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{\theta}{2} \right) \, d\phi d\phi \\
&= -\frac{R^2}{4} \sin \theta \, d\phi d\phi \\
G &= \int -\frac{R^2}{4} \sin \theta \, d\phi d\phi = -R^2 \frac{4\pi}{4} = 4\pi R^2
\end{align*}
\]

Normally \( \sigma \) is defined as positive \( db = 1 \, db \) \( . \) This agrees with the actual cross section of the sphere.
all of the particles in \( \text{d} \Omega \text{d} \phi \)
very far away came out in
the solid angle \( \text{d} \phi \text{d} \Omega = \frac{\text{d} \sigma}{\sigma} \)

\[
\text{d} \sigma = -\frac{R^2}{4} \text{d} \ell \\
\frac{R^2}{4} \frac{\text{d} \sigma}{\text{d} \Omega} = \text{differential cross section}
\]

To understand how to interpret this - consider a beam
with flux \( f \)

\[
f = \frac{\text{# particle}}{\text{Area} \cdot \text{time}}
\]

\[
\text{# collisions} = f \cdot \frac{\text{time}}{\text{time}}
\]

\[
\text{# collisions in solid angle} \ \frac{\text{d} \Omega}{\text{time}} = f \frac{\text{d} \sigma}{\text{d} \Omega} \text{d} \Omega
\]
so \( \frac{d\sigma}{d\Omega} \) gives the distribution of particles scattered in each solid angle.

**Rutherford Scattering**

\[
b = \frac{|k|}{m v^2} \cot \left( \frac{q}{2} \right)
\]

\[
db = \frac{|k|}{m v^2} \left( -1 - \frac{\cos^2 \left( \frac{q}{2} \right)}{\sin^2 \left( \frac{q}{2} \right)} \right) = \frac{-|k|}{m v^2} \frac{1}{\sin \left( \frac{q}{2} \right)} d\Omega
\]

\[
d\sigma = |b| db d\Omega = \frac{|k|^2}{m^2 v^4} \frac{1}{\sin^2 \left( \frac{q}{2} \right)} \frac{\cos \left( \frac{q}{2} \right)}{\sin \left( \frac{q}{2} \right)} d\theta d\phi
\]

\[
d\sigma = \frac{|k|^2}{2 m^2 v^4} \frac{1}{\sin \left( \frac{q}{2} \right)} d\Omega
\]

\[
|k|^2 = \frac{\alpha_1 \alpha_2 \alpha_3}{4 \pi \epsilon_0} = \frac{e^2}{4 \pi \epsilon_0}
\]

\[
\frac{d\sigma}{d\Omega} = \frac{e^4}{32 \pi^2 \epsilon_0^3} \frac{1}{m^2 v^4} \frac{1}{\sin^4 \left( \frac{q}{2} \right)}
\]
This quantity is called the Rutherford scattering cross section clearly - by comparing the hard sphere to the Coulomb case, we see that the differential cross section depends on the target mean free path - how far will a particle travel through a medium without a collision.

Consider a system with \( n \) particles / volume, where each has cross section \( \sigma \).

\[
N(\sigma, dx) \quad \# \text{particles scattered in distance } dx
\]

Since this is dimensionless, let

\[
N \sigma = \frac{1}{\lambda} \quad \lambda \text{ mean free path (length)}
\]
\[ \lambda \text{ represents the distance traveled between collisions.} \]

Consider \( f(x) = \frac{\# \text{ particles at } x}{\text{area time}} \)

\[ f(x)A \text{ # particles/time at } x \]

\[ f(x+\Delta x)A = \# \text{ particles/time at } x + \Delta x \]

\[ \frac{f(x) - f(x+\Delta x)}{\Delta x}A = \frac{\# \text{ collisions in } dx}{\text{time}} \]

\[ n \sigma A f(x) \, dx \]

Taking the limit \( \Delta x \to 0 \)

\[ \frac{df}{dx} = n \sigma f(x) \]

\[ = \frac{1}{\lambda} f(x) \]

This gives

\[ f(x) = \frac{f(0)}{e} \]

This represents the flux of particles that have not collided at depth \( x \).