Lecture 14

Last time - orbits of inverse square forces

Orbit

\[ \frac{1}{r} = e \cos (\theta - \omega) \neq 1 \]

\[ l = \frac{J^2}{1k_1m}, \quad \epsilon^2 = 1 + \frac{2E}{l\epsilon} \]

\[ E > 0 \quad \epsilon^2 > 1 \]

\[ E < 0 \quad \epsilon^2 < 1 \]

\[ \epsilon^2 = 1, \quad E = 0 \]

\[ \epsilon^2 = 0, \quad E = -\frac{l\epsilon}{2\epsilon} \]

Energy at minimum of effective potential

Standard form (\( \theta_0 = 0, \quad \phi = 0 \) smallest \( r \))

\[ l^2 = y^2 + (1-e^2)x^2 + 2Ex \epsilon \]

\[ \epsilon^2 = 1 \]

\[ l^2 = y^2 + 2E \epsilon \]

\[ y^2 = l(e-2x) \]

\[ e^2 > 1 \]

\[ 1 = \frac{1}{a^2} (x+ea)^2 - \frac{y^2}{b^2} \]

\[ a = \frac{l\epsilon}{2E} \]

\[ e^2 < 1 \]

\[ 1 = \frac{1}{a^2} (x-ea)^2 + \frac{y^2}{b^2} \]

\[ b^2 = \frac{l\epsilon E}{2E} = \frac{J^2}{2mE_1} \]

Note: \( a \) only depends on the energy \( E \) and \( b \) depends on \( J^2 \) and \( E \).
Hyberbolic motion and scattering

This requires $e^2 > 1$

\[ \frac{e}{r} = e \cos(\theta) = 1 \quad -k > 0 \]
\[ +k < 0 \]

As $r \to \infty$ the right hand side must vanish. We requires

$e \cos \omega = \pm 1$

$\cos \omega = \pm \frac{1}{e} \quad (\text{for } e > 1 \quad \frac{1}{e} < 1)$

The diagonal lines are limiting bounds of the trajectory. Since $r > 0$ we must have $e \cos \theta = 1 > 0$ - this is true for angles smaller than $\cos^{-1}\left(\frac{1}{e}\right)$. 
the scattering angle $\phi$ is the angular change in direction due to the scattering a long distance away.

For either case $k > 0$ or $k < 0$

$$\phi = \pi - 2\theta$$

we can express $\phi$ in terms of parameters of the problem

$$\theta = \frac{\pi - \phi}{2}$$

$$\cos \theta = \pm \frac{1}{e} = \cos \left( \frac{\pi - \phi}{2} \right)$$

using

$$\cos(A - B) = \cos(A) \cos(B) + \sin(A) \sin(B)$$

$$\cos\left(\frac{\pi - \phi}{2}\right) = \cos\left(\frac{\pi}{2}\right) \cos\left(\frac{\phi}{2}\right) + \sin\left(\frac{\pi}{2}\right) \sin\left(\frac{\phi}{2}\right)$$

$$= \sin\left(\frac{\phi}{2}\right)$$

$$\pm \frac{1}{e} = \sin\left(\frac{\phi}{2}\right)$$

squaring this gives

$$\frac{1}{e^2} = \sin^2\left(\frac{\phi}{2}\right)$$

$$e^2 = \frac{1}{\sin^2\left(\frac{\phi}{2}\right)}$$

$$e^2 - 1 = \frac{1}{\sin^2\left(\frac{\phi}{2}\right)} - 1$$
\[ e^2 - 1 = \frac{1 - \sin^2 \left( \frac{\theta}{2} \right)}{\sin^2 \left( \frac{\theta}{2} \right)} \geq \frac{\cos^2 \left( \frac{\theta}{2} \right)}{\sin^2 \left( \frac{\theta}{2} \right)} \]
\[ = \frac{2E_0}{1k_1} = \frac{b}{a} = \frac{ba}{a^2} = \frac{b^2}{a^2} \]

In both cases \( E > 0 \) in this problem

\[ b^2 = a^2 \cot \left( \frac{\theta}{2} \right) \]

Note
\[ b^2 = \frac{J^2}{2mE} \]

In scattering \( E = \frac{1}{2} mv^2 \) \( J = mbv \)

\[ b^2 = \frac{m^2 b_1^2 v^2}{2m \frac{1}{2} mv^2} = b_1^2 \]

This gives an expression for the impact parameter in terms of the scattering angle

\[ b = \frac{1k_1}{21E_1 \cot \left( \frac{\phi}{2} \right)} = \frac{1k_1}{mv_2 \cot \left( \frac{\phi}{2} \right)} \]
\textbf{scattering}

\textbf{Flux} \ \Phi = \# \text{ particles crossing per unit area / time}

\textbf{Area}

\# scattered per unit time

\[ \frac{dN_{sc}}{dt} = \Phi \sigma \]

\( \sigma \) is called the cross section.

\textbf{scattering from a hard sphere of radius} \( R \)

\begin{align*}
\text{impact parameter} & \quad b = R \sin \alpha \\
\text{scattering angle} & \quad \phi = \pi - 2\alpha \\
& \quad = R \sin \left( \frac{\pi - \phi}{2} \right) \\
\sin (\alpha - \beta) & = \sin (\alpha) \cos (\beta) - \sin (\beta) \cos (\alpha) \\
& = \sin \left( \frac{\phi}{2} \right) \cos \left( \frac{\phi}{2} \right) - \sin \left( \frac{\phi}{2} \right) \cos \left( \frac{\pi}{2} \right) \\
& = \cos \left( \frac{\phi}{2} \right)
\end{align*}
\[ b = R \cos \left( \frac{\phi}{2} \right) \]

\[ \oint \oint 2\pi \int_{0}^{\hbar} d\nu d\lambda \]

\[ db = -\frac{1}{2} R \sin \left( \frac{\phi}{2} \right) \]

\[ dc = kb \, db \, d\phi \]

\[ b = R \cos \left( \frac{\phi}{2} \right) \]

\[ db = -\frac{1}{2} R \sin \left( \frac{\phi}{2} \right) \]

\[ dc = kb \, db \, d\phi = -\frac{1}{4} R^2 \sin \left( \frac{\phi}{2} \right) \cos \left( \frac{\phi}{2} \right) \, d\phi \, d\psi \]

\[ = -\frac{1}{4} R^2 \sin \phi \, d\phi \, d\psi \]

Integrating \( \sin \phi \, d\phi \, d\psi \) gives the angular area of a sphere \( 4\pi \)

\[ \int dc = -\frac{1}{4} R^2 \cdot 4\pi = \pi R^2 \]

This is exactly the cross sectional area of a hard sphere.
interpretation

\[ \frac{d\sigma}{d\Omega} = \sin \phi \, dx \]

note: \( d\sigma = \int d\Omega \, dx = \frac{d\sigma}{d\Omega} \cdot \sin \phi \, dx = \frac{d\sigma}{d\Omega} \sin \phi \, dx \)

The quantity \( \frac{d\sigma}{d\Omega} \) is called the differential cross section.

\[ \int d\sigma = \int \frac{d\sigma}{d\Omega} \, d\Omega = \]

This gives the angular distribution of the scattered particles.

Rutherford scattering

case

hard sphere \( \frac{d\sigma}{d\Omega} = \frac{1}{4} R^2 \)

(by def the cross section is positive — we ignore the — sign)
inverse square

\[ b = \frac{|k|}{mv^2} \cot \left( \frac{\theta}{2} \right) \]
\[ \frac{db}{d\phi} = \frac{|k|}{mv^2} \left( -\frac{\sin \left( \frac{\phi}{2} \right)}{\sin^2 \left( \frac{\phi}{2} \right)} - \frac{\cos \left( \frac{\phi}{2} \right)}{\sin \left( \frac{\phi}{2} \right)} \right) \cdot \frac{1}{2} \, d\phi \]
\[ = \frac{|k|}{2mv^2} \cdot \frac{1}{\sin^2 \left( \frac{\phi}{2} \right)} \, d\phi \]

\( b \, d\phi \, d\chi = \frac{|k|^2}{2m^2 v^4} \cdot \frac{\cos \left( \frac{\phi}{2} \right)}{\sin^3 \left( \frac{\phi}{2} \right)} \, d\phi \, d\chi = \]
\[ = \frac{|k|^2}{2m^2 v^4} \cdot \frac{1}{\sin^4 \left( \frac{\phi}{2} \right)} \cdot \sin \left( \frac{\phi}{2} \right) \cos \left( \frac{\phi}{2} \right) \, d\phi \, d\chi \]
\[ = \frac{|k|^2}{4m^2 v^4} \cdot \frac{1}{\sin^2 \left( \frac{\phi}{2} \right)} \, d\Omega \]

This gives

\[ \frac{d\sigma}{d\Omega} = \frac{|k|^2}{4m^2 v^4} \cdot \frac{1}{\sin^2 \left( \frac{\phi}{2} \right)} \]
\[ R = \frac{e^2}{4\pi \epsilon_0} \]

\[ \frac{d\sigma}{d\Omega} = \left( \frac{e^2}{8\pi \epsilon_0} \right)^2 \cdot \frac{1}{m^2 v^4} \cdot \frac{1}{\sin^2 \left( \frac{\phi}{2} \right)} \]

Note that by looking at the angular distribution it is possible to distinguish a hard sphere from scattering of charged particles.
cross sections and attenuation

How far does a beam of particles extend into a medium consisting of \( n \) particles per unit volume with cross section \( \sigma \)?

\[
f = \text{# particles incident per unit area per unit time}
\]

We assume that each time an incoming particle scatters it is removed from the beam.

Cross section in \( dx \)

\[
n \sigma \, dx = \# \text{ of particles colliding in } dx
\]

\[
n \sigma = \frac{\# \text{ collisions}}{\text{length}}
\]

\[
\frac{1}{n \sigma} = \frac{\text{length}}{\text{collision}} \equiv \lambda
\]

= mean free path
\[ f(x)A = \# \text{ particles} \text{ / time at } x \]
\[ f(x+dx)A = \# \text{ particles} \text{ / time at } x+dx \]

\[ (f(y)-f(x+dx))A = \# \text{ particles scattered} \text{ in length } dx \]

\[ = (nA dx)(f(y)) \]

\[ \frac{\# \text{ particles}}{dx \times A} \text{ incident} \times \frac{\# \text{ particles}}{\text{per unit time that scatter from each target}} \]

Taking the limit \( dx \to 0 \):

\[ -\frac{df}{dx} = n \sigma f dx = \frac{1}{\lambda} f \]

\[ \frac{df}{dx} = -\frac{1}{\lambda} f \]

\[ f(x) = e^{-\frac{x}{\lambda}} f(0) \]

This tells how many particles per unit area / unit time travel a distance \( x \) without scattering.
Rotating coordinate systems

While we have discussed Newton's laws in inertial coordinate systems - sometimes it is useful to use non-inertial coordinate systems.

Consider a solid body (like the earth) rotating about a vector \( \hat{n} \) with angular speed \( \omega \).
We assume that \( \hat{n} \) is given by the right hand rule.

\[ \vec{\omega} = \hat{n} \omega \]

Note that

\( \vec{\omega} \times \vec{F} \)

is \( I \) to \( \vec{F} \) and \( \vec{\omega} \), the magnitude is \( \rho \omega \) which is the velocity.

\[ \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{F} \]
This is true for any vector $\mathbf{F}$
that is at rest in the
body fixed system—in particular
it holds for the unit vectors
in the body fixed system

\[
\frac{d\hat{b}}{dt} = \mathbf{\bar{\omega}} \times \hat{b} \\
\frac{d\hat{j}}{dt} = \mathbf{\bar{\omega}} \times \hat{j} \\
\frac{d\hat{k}}{dt} = \mathbf{\bar{\omega}} \times \hat{k}
\]

A general vector in the body
fixed coordinate system is

\[
\mathbf{\bar{a}} = a_x \hat{b} + a_y \hat{j} + a_z \hat{k}
\]

In the body fixed coordinate
system

\[
\left( \frac{d\mathbf{\bar{a}}}{dt} \right)_b = \left( \frac{da_x}{dt} \right)_b \hat{b} + \left( \frac{da_y}{dt} \right)_b \hat{j} + \left( \frac{da_z}{dt} \right)_b \hat{k}
\]

The components $a_y = \mathbf{\bar{a}} \cdot \hat{j}$ are
rotationally invariant scalas.
If we calculate the time dependence of this same quantity in the inertial coordinate system the body fixed unit vectors change

\[
\begin{align*}
\left( \frac{d\vec{\omega}}{dt} \right)_I &= \left( \frac{d\vec{\omega}}{dt} \right)_b + a_x (\vec{\omega} \times \vec{b}_b) + a_y (\vec{\omega} \times \vec{b}_b) + a_z (\vec{\omega} \times \vec{b}_b) \\
\left( \frac{d\vec{\omega}}{dt} \right)_I &= \left( \frac{d\vec{\omega}}{dt} \right)_b + \vec{\omega} \times \vec{\omega}
\end{align*}
\]

This shows that the velocity of a particle in the inertial coordinate system is the sum of the velocity of the particle in the body fixed system plus the velocity of a particle at \( \vec{r} \) in the body.