Lecture 2

Outline

- Conservative force
- Conservation of energy
- Oscillators

The physics that will be discussed will be limited to 1 dimensional problems. For these problems, Newton's second law for a single particle has the form

\[ m \frac{d^2x}{dt^2} = F(x) \]

(In principle the force could depend on time - but for this problem we consider a force that does not depend on t.)

We start by defining the kinetic energy of the particle

\[ T = \frac{1}{2} m \dot{x}^2 \]
we differentiate $T$ with respect to time (we get using the chain rule

$$\frac{d}{dt} T = \frac{d}{dt} \left( \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 \right) = \frac{1}{2} m \frac{d}{dt} \left( \frac{dx}{dt} \right)^2$$

$$= \frac{1}{2} m \left( \frac{d}{dt} \frac{dx}{dt} \right) \frac{dx}{dt}$$

$$= m \frac{dx}{dt} \frac{d^2x}{dt^2}$$

using Newton's second law in the above equation gives

$$\frac{d}{dt} T = \frac{dx}{dt} \left( m \frac{d^2x}{dt^2} \right) = F(x) \frac{dx}{dt} \quad (1)$$

If we define the potential energy

$$V(x) = - \int_{x_0}^{x} F(x') \, dx'$$

In this expression $x_0$ is any point chosen at convenience. This choice has the implication that $V(x_0) = 0$, so $x_0$ is a point where the potential vanishes.
Note that

\[
\frac{dV}{dx} = \lim_{\Delta x \to 0} \left[ -\int_{x_0}^{x+\Delta x} F(x') \, dx' + \int_{x_0}^{x} F(x') \, dx' \right] \frac{1}{\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \left[ -\int_{x}^{x+\Delta x} F(x') \, dx' \right] \frac{1}{\Delta x}
\]

\[F(x') \approx F(x) + \frac{dF}{dx}(x) (x'-x) + \ldots\]

\[F(x) + \frac{dF}{dx}(x) (x'-x)\]

\[
\lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[ -\Delta x F(x) - \frac{\Delta x^2}{2} \frac{dF}{dx}(x) + \ldots \right]
\]

\[= -F(x)\]

so with the potential defined this way

\[F(x) = -\frac{dV}{dx}(x)\]

(this result is independent of \( x_0 \))
using this in equation (1) gives

\[
\frac{dT}{dt} = F(x) \frac{dx}{dt} = (- \frac{dV}{dx} \frac{dx}{dt})
\]

using the chain rule gives

\[
\frac{dT}{dt} = - \frac{dV}{dt}
\]
or

\[
\frac{d}{dt} (T+V) = 0
\]

This means that independent of
the values of \(x(t)\) and \(\dot{x}(t)\) that

\[
E = T+V
\]
is a constant.

The constant is called the energy
of the system.

Unit of energy = joule = 1 kg m^2 s^-2

While \(V\) and \(T\) may change,
the sum remains constant.
while classical mechanics has a limited range of validity—energy conservation is a property that holds even when classical mechanics is valid.

\[ \frac{1}{2} m \dot{x}^2 = E - V(x) \geq 0 \]

When \( E = V(x) \) then \( \ddot{x} = 0 \) at \( x = 0 \)

We can use this to replace Newton's second law by a first-order equation

\[ \dot{x}^2 = \frac{2}{m} (E - V(x)) \]

\[ \frac{dx}{dt} = \pm \sqrt{\frac{2}{m} (E - V(x))} \]
$$\pm \frac{dx}{\sqrt{\frac{2}{m} (E-V(x))}} = dt$$

$$t-t_0 = \pm \int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2}{m} (E-V(x))}}$$

most of this is very general

example: simple pendulum

In this example there are 2 forces. The tension in the string and the gravitational force
\[
\sin \theta = \frac{x}{r} \quad \cos \theta = -\frac{y}{r}
\]

The tension in the string cancels the component of the gravitational force parallel to the string.

The component of the gravitational force perpendicular to the tension causes the motion. This component is

\[
F = -mg \sin \theta
\]

The displacement is in the direction of the arc length \(d\theta\)

\[
V = -\int_{\theta_0}^{\theta} Fds = -\int_{\theta_0}^{\theta} Fd\theta
\]

\[
= mgL \int_{\theta_0}^{\theta} \sin \theta d\theta
\]

\[
= mgL \left[ -\cos \theta \right]_{\theta_0}^{\theta}
\]

\[
= -mgL \cos \theta + mgL \cos \theta_0
\]

\(\theta_0\) is arbitrary -

\[
V = mgL (\cos \theta - \cos \theta_0)
\]

We can also calculate the kinetic energy.
\[ T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \]

\[ x = l \sin \theta \quad \dot{x} = l \cos \theta \dot{\theta} \]
\[ y = -l \cos \theta \quad \dot{y} = -l (-\sin \theta) \dot{\theta} = l \sin \theta \dot{\theta} \]

\[ T = \frac{1}{2} m \left( l^2 \cos^2 \theta \dot{\theta}^2 + l^2 \sin^2 \theta \dot{\theta}^2 \right) \]
\[ = \frac{m l^2}{2} (\cos^2 \theta + \sin^2 \theta) \dot{\theta}^2 \]
\[ = \frac{m l^2}{2} \dot{\theta}^2 \]

The energy is

\[ E = T + V = \frac{m l^2}{2} \dot{\theta}^2 + m g l (\cos \theta - \cos \theta_0) \]

The minimum of the potential energy is when
\[ \frac{\partial V}{\partial \theta} = 0 \rightarrow m g l \sin \theta = 0 \]
This is when \( \theta = 0 \)

\[ E = \frac{m l^2}{2} \dot{\theta}^2 + m g l (1 - \cos \theta_0) \]

\[ m l^2 \dot{\theta}^2 = 2 (E - m g l + m g l \cos \theta_0) > 0 \]

The allowed values of \( \cos \theta_0 \) require that the right side is non-negative

\[ \dot{\theta}^2 = 0 \text{ if } E - m g l + m g l \cos \theta_0 = 0 \]
If $E > 2mgR$ then the ball never stops - instead it spins around the origin. Otherwise we define

$$mgR \cos \theta_0 = mgR - E$$

$$\ell^2 \dot{\theta}^2 = 2g \ell (\cos \theta - \cos \theta_0)$$

$$\dot{\theta} = \pm \sqrt{\frac{2g}{\ell} \sqrt{\cos \theta_0 - \cos \theta_0}}$$

$\theta_0$ is called a classical turning point. When $\theta = \pm \theta_0$ the pendulum stops and changes direction.

Is $\theta$ maximal when $\dot{\theta} = 0$?

$$\dot{\theta}_{\text{max}} = \sqrt{\frac{2g}{\ell} \sqrt{1 - \cos \theta_0}}$$
For a particle in one dimension the force on the particle is

\[ F = - \frac{dV}{dx}(x) \]

where \( V(x) \) is the potential. The force vanishes when

\[ \frac{dV}{dx}(x) = 0 \]

At \( x_0 \), where \( \frac{dV}{dx}(x) = 0 \) is called an equilibrium point. At that point

\[ m \frac{d^2x}{dt^2} = 0 \]

so if the particle is at rest it will stay at \( x_0 \).

If we consider small displacements from equilibrium we can approximate \( V(x) \) near \( x_0 \) using a Taylor series

\[ V(x) = V(x_0) + \frac{dV}{dx}(x_0)(x-x_0) + \frac{1}{2} \left( \frac{d^2V}{dx^2}(x_0) \right)(x-x_0)^2 + \cdots \]

which vanishes and becomes a constant at equilibrium

\[ V(x) \approx \frac{1}{2} \left| \frac{d^2V}{dx^2}(x_0) \right| (x-x_0)^2 \]
If \( \frac{d^2 V}{dx^2}(x) > 0 \) then the potential energy increases for small displacements from equilibrium.

If \( \frac{d^2 V}{dx^2}(x) < 0 \) then the potential energy decreases for small displacements from equilibrium.

The equilibrium is called stable if \( \frac{d^2 V}{dx^2}(x) > 0 \) \( (\text{-} \frac{dV}{dx} < 0 \text{ for } x \text{ near } x_0) \) (this gives a restoring force.)

The equilibrium is called unstable if \( \frac{d^2 V}{dx^2}(x) < 0 \) \( (\text{-} \frac{dV}{dx} > 0 \text{ for } x \text{ near } x_0) \) (this gives an accelerating force.)

In the stable case we define

\[
K = \frac{d^2 V}{dx^2}(x) > 0
\]
For small displacements about a stable equilibrium

\[ E = \frac{1}{2} m x^2 + V(x) + \frac{1}{2} k (x - x_0)^2 \]

we can change the origin of the coordinate system to

\[ x' = x - x_0 \]

\[ \frac{dx'}{dt} = \frac{dx}{dt} \quad V(x) \rightarrow V(x - x_0) \quad k = \frac{d^2 V}{dx'^2} > 0 \]

This equation becomes

\[ E - V(0) = \frac{1}{2} m x'^2 + \frac{1}{2} k x^2 \]

we redefine the potential so \( V(0) = 0 \)

\[ E = \frac{1}{2} m x'^2 + \frac{1}{2} k x^2 \]

This is the energy of a simple harmonic oscillator.

There are 2 ways to solve this - we could use energy conservation

\[ x^2 = \frac{2}{m} \left( E - \frac{1}{2} k x^2 \right) \]

\[ \frac{dx}{dt} = \sqrt{\frac{2}{m} \left( E - \frac{1}{2} k x^2 \right)} \]

\[ t - t_0 = \int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2}{m} \left( E - \frac{1}{2} k x^2 \right)}} \]
The other method involves using Newton's second law

\[ m \dddot{x} = F = -\frac{dV}{dx} = -\left( \frac{d}{dx} \left( \frac{1}{2} k x^2 \right) \right) = -kx \]

This is a second order differential equation with constant coefficients.

To solve it we write it as a pair of equations:

\[ \dot{x} = y \]
\[ \dot{y} = x = -\frac{k}{m} x \]

We write this as a matrix equation

\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix} x \\ v \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ \frac{-k}{m} & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \\
\frac{d^2}{dt^2} \begin{pmatrix} x \\ v \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ \frac{-k}{m} & 0 \end{pmatrix} \frac{d}{dt} \begin{pmatrix} x \\ v \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ \frac{-k}{m} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{-k}{m} & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ \frac{-k}{m} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{-k}{m} & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}
\end{align*}
\]
\[
\frac{d^{2n}}{dt^{2n}} \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix}^{2n} \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} -\frac{k}{m} & 0 \\ 0 & -\frac{k}{m} \end{pmatrix}^{n} \begin{pmatrix} x \\ v \end{pmatrix} = (-1)^n \begin{pmatrix} (\frac{k}{m})^n & 0 \\ 0 & (\frac{k}{m})^n \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}
\]

Similarly,
\[
\frac{d^{2n+1}}{dt^{2n+1}} \begin{pmatrix} x \\ v \end{pmatrix} = (-1)^n \begin{pmatrix} (\frac{k}{m})^n & 0 \\ 0 & (\frac{k}{m})^n \end{pmatrix} \frac{d}{dt} \begin{pmatrix} x \\ v \end{pmatrix} = (-1)^n \begin{pmatrix} (\frac{k}{m})^n & 0 \\ 0 & (\frac{k}{m})^n \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} = (-1)^n \begin{pmatrix} (\frac{k}{m})^n & 0 \\ 0 & (\frac{k}{m})^n \end{pmatrix} \begin{pmatrix} x \\ -\frac{k}{m} x \end{pmatrix}
\]

We can use these expressions to construct the Taylor series about \( t=0 \)
\[
\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n}{dt^n} \begin{pmatrix} x \\ v \end{pmatrix}^{(0)} t^n = \\
\sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^{2n}}{dt^{2n}} \begin{pmatrix} x \\ v \end{pmatrix}^{(0)} t^{2n} + \\
\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \frac{d^{2n+1}}{dt^{2n+1}} \begin{pmatrix} x \\ v \end{pmatrix}^{(0)} t^{2n+1}
\]

Using the above expressions
\[
\begin{align*}
(X(t)) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left( \sqrt{\frac{k}{m}} t \right)^{2n} \left( X(0) \right) + \\
V(t) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left( \sqrt{\frac{k}{m}} t \right)^{2n+1} \sqrt{\frac{m}{k}} \left( V(0) \right)
\end{align*}
\]

The series are Taylor series for the sine and cosine:

\[
\begin{align*}
X(t) &= \cos(\sqrt{\frac{k}{m}} t) X(0) + \sqrt{\frac{m}{k}} \sin(\sqrt{\frac{k}{m}} t) V(0) \\
V(t) &= \cos(\sqrt{\frac{k}{m}} t) V(0) - \sqrt{\frac{k}{m}} \sin(\sqrt{\frac{k}{m}} t) X(0)
\end{align*}
\]

It is easy to check that this is the solution.

There are 2 constants \(X(0)\), \(V(0)\) and 2 independent solutions of the equation

\[
\sin(\sqrt{\frac{k}{m}} t) \cos(\sqrt{\frac{k}{m}} t)
\]

This method for solving differential equations will be used later when larger matrices are involved.
example - small amplitude oscillations

\[\begin{align*}
q_1 & = q_2 = q \\
-a & \\
l' &= \frac{q^2}{4\pi \varepsilon_0} \frac{1}{(x+a)^2} - \frac{q^2}{4\pi \varepsilon_0} \frac{1}{(x-a)^2} \\
V &= \frac{q^2}{4\pi \varepsilon_0} \frac{1}{(x+a)} + \frac{q^2}{4\pi \varepsilon_0} \frac{1}{(a-x)}
\end{align*}\]

The force vanishes when \( x = a \)

\[V = \frac{q^2}{4\pi \varepsilon_0 \frac{2a}{a^2-x^2}}\]

expanding about \( x = 0 \)

\[V = \frac{q^2}{4\pi \varepsilon_0} \frac{2a}{a^2} + \frac{q^2}{4\pi \varepsilon_0} \frac{2a - 2x}{(a^2-x^2)^\frac{1}{2}} x + \frac{q^2}{4\pi \varepsilon_0} \left( \sum_{n=1}^{\infty} \frac{4x}{(a^2-x^2)^n} \right) \frac{1}{2} x^2\]

\[\sum_{n=1}^{\infty} \frac{4x}{(a^2-x^2)^n} = \frac{q^2}{4\pi \varepsilon_0 a^3}\]

This is the spring constant - the angular frequency is

\[\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{q^2}{\pi \varepsilon_0 ma^3}}\]
The simplest way to solve
\[ m \frac{dx}{dt} = -kx \]
is to assume solutions of the form
\[ x = A \sin \omega t + B \cos \omega t \]
\[ \omega = \sqrt{\frac{k}{m}} \]
\[ x(0) = A + B \]
\[ \frac{dx}{dt}(0) = \omega A - \omega B \cdot 0 \]

This gives
\[ B = x(0) \quad B = \frac{1}{\omega} \dot{x}(0) \]
\[ x(t) = \frac{\dot{x}(0)}{\omega} \sin \omega t + x(0) \cos \omega t \]

This agrees with the series solution.

Consider
\[ e^{i \omega t} = \sum \frac{1}{n!} (i)^n (\omega t)^n = \]
\[ = \sum \frac{1}{(2n)!} (-1)^n (\omega t)^{2n} + i \sum \frac{1}{(2n+1)!} (-1)^n (\omega t)^{2n+1} \]
\[ = \cos(\omega t) + i \sin(\omega t) \]
taking complex conjugates

\[ e^{-i\omega t} = \cos \omega t - i \sin \omega t \]

adding

\[ \cos (\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2} \]
\[ \sin (\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i} \]

clearly

\[ \frac{d^2}{dt^2} e^{i\omega t} = (\pm i\omega)^2 e^{i\omega t} = -\frac{k}{m} e^{i\omega t} \]

we see that we could have expressed the solution

\[ x(t) = A e^{i\omega t} + B e^{-i\omega t} \]
\[ = (A + B) \cos \omega t + i(A - B) \sin \omega t \]
\[ \frac{x(t)}{x(0)} \]

finally we note that if

\[ \ddot{z} = -\frac{k}{m} z \quad z = x + iy \]
\[ (\ddot{x} + \frac{k}{m} x) + i (\ddot{y} + \frac{k}{m} y) = 0 \]

both the real and imaginary parts of \( z(t) \) are separately solutions.
while conservation of energy
normally hold) for isolated systems,
most macroscopic systems are
not isolated. In this case some of
the energy is transferred to
the system. From the perspective
of the oscillating energy is lost to
heat. In this case the forces
are velocity dependent and tend to
reduce the velocity

\[ M \frac{d^2 x}{dt^2} = -kx - \alpha x \]

To solve this equation we
assume a solution of exponential
form

\[ x(t) = Ae^{\beta t} \]

Inserting this in the equation

\[(mB^2 + k + \alpha B)x(t) = 0\]

this give a quadratic equation

For \( B \)

\[ B = \frac{-\alpha \pm \sqrt{\alpha^2 - 4mk}}{2m} \]

\[ = -\frac{\alpha}{2m} \pm i\frac{\sqrt{k^2 - \alpha^2}}{4m^2} \]
In this case there are still two independent solutions

\[ x(t) = e^{-\frac{1}{2\omega t}} \left( \frac{1}{\sqrt{m}} \sqrt{1 - \frac{\omega^2}{4\omega^2} t} + \frac{i}{\sqrt{m}} \sqrt{1 - \frac{\omega^2}{4\omega^2} t} \right) \]

\[ x(t) = e^{-\frac{1}{2\omega t}} \left( A' \cos \left( \sqrt{\frac{k}{m}} \sqrt{1 - \frac{\omega^2}{4\omega^2} t} \right) + B' \sin \left( \sqrt{\frac{k}{m}} \sqrt{1 - \frac{\omega^2}{4\omega^2} t} \right) \right) \]

Here the solution is a product of an oscillating solution and one that decays exponentially.

If \( x^2 \frac{1}{4\omega^2} \geq 1 \) then there are 2 negative roots - giving decay with no oscillations.

If \( x^2 \frac{1}{4\omega^2} = 1 \) the system is called critically damped. It simply decays with no oscillation.