Lecture 20
Homework ass: q 11 16 20 ch6: 1, 2

Last time we discussed potentials far from the source

$$\Phi (r) = k \int \frac{\rho(r')}{|r-r'|} d^3r'$$

where

- $k = \frac{1}{4\pi \epsilon_0}$ for the electric potential
- $k = -G$ for the gravitational potential

for $r \gg r'$

$$\Phi = \frac{k_0}{r} \int \rho(r') d^3r' + \frac{k s}{r^3} \int \vec{r}' \rho(r') d^3r'$$

$$+ \sum \frac{k r_{ij} r_j}{r^5} \left( \frac{1}{2} \int (3 \vec{r}'_i r'_j - s_{ij} r'_i) \rho(r') d^3r' \right) + \ldots$$

This is called the multipole expansion

- $\int \rho(r') d^3r = \text{charge or mass}$
- $\int \vec{r}' \rho(r') d^3r = \vec{p} = \text{dipole moment}$
- $\int (3 \vec{r}'_i r'_j - s_{ij} r'_i) \rho(r') d^3r' = Q_{ij}$
  quadrapole tensor.
alot of information about the
can be determined by measuring
some constants

\[ M, a, \bar{F}, \Omega \]

These are the first three terms
in an infinite expansion. It
is possible to construct \( \rho(\vec{r}) \)
given all of the constants in
the expansion.

Given the constants above be
get a rough idea about the
key features of \( \rho(r) \) without
knowing \( \rho(r) \).

To use these formulas, we use
experimental information to
determine \( \bar{F}, \Omega \). Then

\[ \bar{F} = - \vec{\nabla} \left( - m \Phi_0 (\vec{r}) \right) \\
- \vec{\nabla} \left( q \Phi_e (\vec{r}) \right) \]

We assume that we do not know
\( \rho(\vec{r}) \). If \( \rho(r) \) is known then
\( \Phi(r) \) can be computed either
by integration or by solving
a differential equation.
Integration method

\[ \Phi(\vec{r}) = k \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} \, d^3r' \]

Example: Assume \( \rho(\vec{r}) = \rho(r) \) where \( r \) is the distance from the origin (no angle dependence)

\[ \Phi(\vec{r}) = k \int \frac{\rho(r')}{r' \sqrt{r^2 - 2r'r\cos\theta + r'^2}} \, r'^2 \, dr' \sin\theta' \, d\phi' \]

where
\[ \phi: 0 \rightarrow 2\pi \]
\[ \theta: 0 \rightarrow \pi \]
\[ r: 0 \rightarrow \infty \]

Note that
\[ |\vec{r} - \vec{r}'| = \sqrt{r^2 - 2rr'\cos\theta + r'^2} \]

If we choose coordinates so \( \vec{r} \) is in the direction of the \( z' \) axis then
\[ \vec{r} \cdot \vec{r}' = rr' \cos\theta \] where \( \theta \) is the angle between \( \vec{r} \) and \( \vec{r}' \).

\[ \Phi(r) = k \int \frac{\rho(r)}{\sqrt{r^2 - 2rr'\cos\theta + r'^2}} \, r'^2 \, dr' \sin\theta' \, d\phi' \]

Since the integrand is independent of \( \phi' \), the \( \phi' \) integral gives
\[ E(r) = 2\pi k \int_0^r \int_0^{\pi} \frac{\rho(r')}{\sqrt{r^2 - 2rr' \cos \theta + r'^2}} \, d\theta \, dr' \]

To do the \( G' \) integral let \( u = \cos \theta \):

\[ du = -\sin \theta \, d\theta; \quad 1 \to -1 \]

\[ E(r) = 2\pi k \int_0^r r' \, dr' \int_{-1}^{1} (-du) \frac{\rho(r')}{\sqrt{r^2 + r'^2 - 2rr' u}} = \]

\[ = 2\pi k \int_0^r r' \, dr' \int_{-1}^{1} \left( -\frac{1}{r'} \right) \frac{\rho(r')}{\sqrt{r^2 + r'^2 - 2rr' u}} \, du \]

\[ = 2\pi k \int_0^r r' \, dr' \left( -\rho(r') \left[ \sqrt{(r-r')^2 - \sqrt{(r+r')^2}} \right] \right) \]

\[ = \frac{2\pi k}{r} \int_0^r r' \, dr' \rho(r') \left( 1r+r' - 1r-r' \right) \]

at this point we need to consider two cases:

1. \( r > R \)

\[ |r+r'| - |r-r'| = r+r'-r+r' = 2r' \]

In this case the integral becomes

\[ = \frac{4\pi k}{r} \int_0^r r' \rho(r') \, dr' \]

\[ = \frac{k}{r} \int \rho(r') \, r'^2 \sin \theta \, d\theta \, dr' \]

In this case the integral is

\[ M \quad \text{for} \quad k = -\frac{1}{4\pi} \]
which gives

\[ \Phi = -\frac{GM}{r} \quad \text{and} \quad \Phi = \frac{Q}{4\pi\varepsilon_0} \frac{1}{r} \]

These results are exact; the only assumption that was used is that \( \rho \) is independent of \( \theta, \phi \).

2. The second case is when the particle is inside of the mass or charge distribution - then \( r < R \).

In this case,

\[ \Phi(r) = \frac{2\pi \rho R}{r} \int_0^r r' \rho(r') \, 2\pi r' \, dr' \quad (r' < r) \]

\[ + \frac{2\pi \rho R}{r} \int_r^R r' \rho(r') \, 2\pi r' \, dr' \quad (r' > r) \]

\[ = \frac{4\pi \rho R}{r} \int_0^r \rho(r') r'^2 \, dr' + 4\pi \rho \int_r^R r' \rho(r') \, dr' \]

To go further it is necessary to assume a form for \( \rho(r') \). If \( \rho = \text{constant} \) (uniform density)

\[ = \frac{4\pi \rho R}{r} \frac{r^3}{3} + 4\pi \rho \left( \frac{R^3}{2} - \frac{r^3}{2} \right) + 4\pi \rho kr^2 \left( \frac{1}{3} - \frac{1}{2} \right) + 2\pi \rho R^2 \]
The second term is a constant—it does not contribute to the force. The first term is

$$\Phi(r) = - \frac{4\pi \rho kr^4}{6} = - \frac{2}{3} \pi \rho k r^2$$

when $k = -6$

$$\Phi = \frac{2}{3} \pi \rho G r^2$$

In this case the potential energy is

$$V = m \Phi = \frac{2}{3} m \pi \rho G r^2$$

$$F_r = - \frac{\partial V}{\partial r} = - \frac{4}{3} m \pi \rho G r$$

which is a linear restoring force like a spring.

This means that if you made an infinitely deep well through the center of the earth and dumped in you would execute simple harmonic motion.

The speed at the center of the earth would be

$$\frac{1}{2} m V^2 = \frac{2}{3} m \pi \rho G R^2$$

$$V^2 = \left( \frac{4}{3} \pi \rho \right) G m R^2 = 6m_e R_e^2$$

$$V = \sqrt{6m_e R_e^2}$$
(This assumes no dissipation and non-relativistic mechanics are valid.)

The second approach to computing the charge density is by solving a differential equation. We illustrate this using the example of an electric field.

Consider a charge at the origin

\[ \vec{F} = \frac{Q}{4\pi\varepsilon_0} \frac{\vec{r}}{r^3} \]

\[ \vec{E} = \frac{\vec{F}}{q} = \frac{Q}{4\pi\varepsilon_0} \frac{\vec{r}}{r^3} \]

If we integrate over a surface of radius \( R \)

\[ \int \vec{E} \cdot \hat{r} \, ds = \int \vec{E} \cdot \hat{r} \, R^2 \sin\theta \, d\theta \, d\phi \]

\[ = \frac{Q}{4\pi\varepsilon_0} \int \frac{1}{R^3} \, R^2 \sin\theta \, d\theta \, d\phi = \frac{4\pi}{4\pi\varepsilon_0} = \frac{Q}{\varepsilon_0} \]

This is the simplest example of a surface integral

\[ \int_S \vec{E} \cdot \hat{n} \, ds \]
\[ dS \cdot \hat{n} \cdot \vec{F} = \text{is the projection of the surface element } dS \text{ on the plane } \perp \text{ to } \hat{n} \text{ which is } r^2 \times \text{solid angle} \]
\[ dS \cdot \cos \theta = r^2 \, d\Omega \quad d\Omega \text{ is the solid angle.} \]

\[ \int E \cdot \hat{n} \, dS = \frac{1}{4\pi} \int \frac{G}{r^2} \, r^2 \, d\Omega = \frac{G}{4\pi \varepsilon_0} 4\pi = \frac{G}{\varepsilon_0}. \]

If the charge is outside of the surface the normal and \( \hat{n} \) are in opposite directions and the contribution from the solid angle cancels the term on the opposite side.

In general we get Gauss law

\[ \int E \cdot \hat{n} \, dS = \frac{Q}{\varepsilon_0}, \]

where
The integral is over a closed surface \( \Gamma \) is inside of the closed surface.

If we have several changes

\[
\int \mathbf{E} \cdot \mathbf{n} \, ds = \oint -\nabla \Phi \cdot \mathbf{n} \, ds =
\int \mathbf{E} \cdot \mathbf{n} \, ds = \frac{q_i}{\varepsilon_0} = \frac{Q}{\varepsilon_0}
\]

where \( Q \) is the net charge enclosed by the surface.

For a continuous change this becomes:

\[
\int \mathbf{E} \cdot \mathbf{n} \, ds = \frac{1}{\varepsilon_0} \int \rho(r) \, d^3r
\]

The next step is to use the divergence theorem.

Consider a cube of side length \( a \).

\[
\int \mathbf{E} \cdot \mathbf{n} \, ds =
\]

\[
\begin{align*}
\int_0^a dx \int_0^a dy \int_0^a dz & \left( -E_x(0, y, z) + E_x(a, y, z) \right) + \\
\int_0^a dx \int_0^a dy \int_0^a dz & \left( -E_y(x, 0, z) + E_y(x, a, z) \right) + \\
\int_0^a dx \int_0^a dy \int_0^a dz & \left( -E_z(x, y, 0) + E_z(x, y, a) \right)
\end{align*}
\]
we can write

\[-E_x(0yz) + E_x(ayz) = \]
\[\int_0^a \frac{d}{dx} E_x(xyz) \]

so

\[\int_0^a \int_0^a \int_0^a (-E_x(0yz) + E_x(ayz)) = \]
\[\int_0^a \int_0^a \int_0^a \int_0^a \frac{dE_x}{dx} \]

summing up the contributions from the other 4 surfaces, gives:

\[\int S \mathbf{E} \cdot \mathbf{n} \, dS = \int V \mathbf{D} \cdot \mathbf{E} \, d\mathbf{v} \]

This is true for any cube.

If we break a volume into many small cubes and add the contributions from each cube.

The surface integrals on adjacent surfaces cancel because the outward normals have opposite directions.

What remains are the normals on the boundary of the volume.
In the limit that the cube size vanishes we get

\[ \int_S \mathbf{E} \cdot \mathbf{n} \, ds = \int_V \nabla \cdot \mathbf{E} \, dv \]

for any surface - this is called the divergence theorem.

\[ \int_V \nabla \cdot \mathbf{E} \, dv = \int_S \mathbf{E} \cdot \mathbf{n} \, ds = \frac{1}{\varepsilon_0} \int_V \rho \, dv \]

or

\[ \int_V \left( \nabla \cdot \mathbf{E} - \frac{1}{\varepsilon_0} \rho \right) \, dv = 0 \]

For this to be true in any volume we must have

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \]

This is called the differential form of Gauss's law.

Using

\[ \nabla \cdot \mathbf{E} = \nabla (-\nabla \Phi) = -\nabla^2 \Phi \quad \text{given} \]

\[ \nabla^2 \Phi = -\frac{\rho}{\varepsilon_0} \]
so given $\rho$ this is a differential equation that determines $\Phi$.

This is called Poisson's equation.

It holds for gravity with $\frac{1}{\epsilon_0} = -4\pi\varepsilon_0$

$$\nabla^2 \Phi = -4\pi\varepsilon_0 \rho.$$

In both cases - if we know the potential then we can find
the potential.

**Summary**

Potential provide a useful means to express the forces on a
particle in a complex system

This can be done using the change density $\rho$ by experimentally
determining multipole moments.
Two body problems

2nd law
\[ m_1 \ddot{\vec{r}}_1 = m_1 \vec{g} + \vec{F}_{12} \]
\[ m_2 \ddot{\vec{r}}_2 = m_2 \vec{g} - \vec{F}_{12} \]

in the absence of gravity
\[ m_1 \ddot{\vec{r}}_1 + m_2 \ddot{\vec{r}}_2 = 0 \]
\[ \frac{d}{dt} \left( m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2 \right) = (m_1 + m_2) \frac{d}{dt} \left[ \frac{m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2}{m_1 + m_2} \right] \]
\[ \vec{R} = \frac{m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2}{m_1 + m_2} = \text{center of mass position} \]
\[ M = m_1 + m_2 = \text{mass of system} \]

then the above equation becomes
\[ M \frac{d\vec{R}}{dt} = 0 \quad M \dot{\vec{R}} = \text{const} \]

and with \( \vec{g} \) back in
\[ M \ddot{\vec{R}} = M \vec{g} \]

It is also useful to define the relative coordinate
\[ \vec{r} = \vec{r}_1 - \vec{r}_2 \]
\[ \vec{r} = \vec{r}_1 - \vec{r}_2 = q + \frac{F_{12}}{m_1} - q + \frac{F_{12}}{m_2} \]
\[ = \left( \frac{1}{m_1} + \frac{1}{m_2} \right) F_{12} \]
\[ = \frac{m_1 m_2}{m_1 + m_2} F_{12} \]

we define the reduced mass
\[ \mu = \frac{m_1 m_2}{m_1 + m_2} \]
\[ \mu \vec{r} = \vec{F} \]

In this case the equations become
\[ \mu \ddot{r} = \vec{F} \]
\[ M \ddot{\vec{r}} = M \vec{g} \]

these are completely equivalent to the original equations

we can solve for \( \vec{F}_1, \vec{F}_2 \) in terms of \( \vec{r} \)

\[ M \vec{r} = m_1 \vec{r}_1 + m_2 \vec{r}_2 \]
\[ m_1 \vec{r} = m_1 \vec{r}_1 - m_1 \vec{r}_2 \]
\[ m_2 \vec{r} = m_2 \vec{r}_1 - m_2 \vec{r}_2 \]
\[ M \vec{r} - m_1 \vec{r} = M \vec{r}_2 \]
\[ M \vec{r} + m_2 \vec{r} = M \vec{r}_1 \]

\[ \vec{r}_1 = \vec{R} + \frac{m_2}{M} \vec{r} \]
\[ \vec{r}_2 = \vec{R} - \frac{m_1}{M} \vec{r} \]
in terms of these variables

\[ \mathbf{J} = m_1 \mathbf{r}_1 \times \dot{\mathbf{r}}_1 + m_2 \mathbf{r}_2 \times \dot{\mathbf{r}}_2 = \]
\[ = m_1 (\mathbf{r} + \frac{m_2}{M} \mathbf{r} \times \dot{\mathbf{r}}) \times (\dot{\mathbf{r}} + \frac{m_1}{M} \dot{\mathbf{r}}) + \]
\[ m_2 (\mathbf{r} - \frac{m_1}{M} \mathbf{r} \times \dot{\mathbf{r}}) \times (\dot{\mathbf{r}} - \frac{m_1}{M} \dot{\mathbf{r}}) = \]
\[ = (m_1 + m_2) \mathbf{r} \times \dot{\mathbf{r}} + \frac{m_1 m_2^2 + m_2 m_1^2}{M^2} \mathbf{r} \times \dot{\mathbf{r}} \]
\[ + \frac{m_1 m_2}{M} \mathbf{r} \times \dot{\mathbf{r}} \]
\[ \left( \frac{m_2 - m_1}{M} \right) \mathbf{F} \times \mathbf{r} \]
\[ \mathbf{M} \mathbf{r} \times \dot{\mathbf{r}} + \frac{m_1 m_2 (m_1 + m_2)}{M^2} \mathbf{r} \times \dot{\mathbf{r}} \]
\[ \mathbf{J} = \mathbf{M} \mathbf{r} \times \dot{\mathbf{r}} + \mathbf{M} \mathbf{F} \times \dot{\mathbf{r}} \]

If we compute the kinetic energy

\[ T = \frac{1}{2} m_1 \dot{\mathbf{r}}_1^2 + \frac{1}{2} m_2 \dot{\mathbf{r}}_2^2 = \]
\[ = \frac{1}{2} m_1 (\dot{\mathbf{r}} + \frac{m_2}{M} \dot{\mathbf{r}} \times \dot{\mathbf{r}})^2 + \frac{1}{2} m_1 (\dot{\mathbf{r}} - \frac{m_1}{M} \dot{\mathbf{r}} \times \dot{\mathbf{r}})^2 \]
\[ = \frac{1}{2} (m_1 + m_2) \dot{\mathbf{r}}^2 \]
\[ + \frac{1}{2} \frac{m_1 m_2 + m_2 m_1}{M^2} \dot{\mathbf{r}} \times \dot{\mathbf{r}}^2 \]
\[ + \frac{1}{2} \frac{2 m_1 m_2}{M} (\mathbf{r} \times \dot{\mathbf{r}} - \dot{\mathbf{r}} \times \mathbf{r}) \]
\[ + \frac{1}{2} \frac{m_1 m_2 + m_2 m_1}{M^2} \dot{\mathbf{r}} \times \dot{\mathbf{r}}^2 \]
\[ T = \frac{1}{2} M \dot{R}^2 + \frac{1}{2} \dot{u} \dot{r}^2 \]

For a conservative force with potential energy \( V(\vec{r}_1, \vec{r}_2) \)

\[ L = T - V \]
\[ = \frac{1}{2} M \dot{R}^2 + \frac{1}{2} \dot{u} \dot{r}^2 - V(\vec{r}) \]

If we include the gravitational force

\[ V_1 = m_1 g z_1 \quad V_2 = m_2 g z_2 \]
\[ = (m_1 + m_2) g \left( \frac{z_1 m_1 + z_2 m_2}{M} \right) \]
\[ = M g z \]

where \( \vec{r} = (x, y, z) \)

\[ L = \frac{1}{2} M \dot{R}^2 + \frac{1}{2} \dot{u} \dot{r}^2 - V(\vec{r}) - M g z \]

Lagrange's equations are

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}_i} \right) - \frac{\partial L}{\partial r_i} = 0 \]
\[ M \ddot{R} = -M g \quad \ddot{g} = g \hat{k} \]
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}_i} \right) - \frac{\partial L}{\partial r_i} = \dot{u} \ddot{r} = -V_r \dot{V} \]

which gives the same equations that were obtained using

Newton's second law
Because the equations for $\bar{\mathbf{R}}(t)$ and $\bar{\mathbf{F}}(t)$ are independent, it is useful to solve the problem in a coordinate system where $\bar{\mathbf{R}}(t) = 0$. (In the case of the uniform gravitational field, this is an accelerated coordinate system—but this choice does not change the $\bar{\mathbf{F}}$ equations.)

In this frame

\[
\ddot{\mathbf{r}} = \bar{\mathbf{F}}(\bar{\mathbf{r}})
\]

\[
\ddot{\mathbf{r}}_1 = \ddot{\mathbf{r}} + \frac{m_1}{M} \ddot{\mathbf{r}}
\]

\[
\ddot{\mathbf{r}}_2 = \ddot{\mathbf{r}} - \frac{m_1}{M} \ddot{\mathbf{r}}
\]

\[
m_1 \ddot{\mathbf{r}}_1 + m_1 \ddot{\mathbf{r}}_2 = \frac{m_1 m_2}{M} (\ddot{\mathbf{r}} - \ddot{\mathbf{r}}) = 0
\]

In this case,

\[
m_1 \ddot{\mathbf{r}}_1 + m_1 \ddot{\mathbf{r}}_2 = 0
\]

\[
m_1 \dot{\mathbf{r}}_1 + m_1 \dot{\mathbf{r}}_2 = M \ddot{\mathbf{r}} = 0
\]

\[
m_1 \dot{\mathbf{r}}_1 = -m_1 \dot{\mathbf{r}}_2
\]

Center of mass coordinate system

$\bar{\mathbf{R}} = \bar{\mathbf{r}} = 0$.
once the problem is solved in the center of mass system the solution in the general coordinate system is obtained by

\[
\begin{align*}
\vec{r}_1(t) &= \vec{R}(t) + \frac{m_2}{M} \vec{r} \quad (t) = \vec{R}(t) + \vec{r}_{1\text{cm}}(t) \\
\vec{r}_2(t) &= \vec{R}(t) - \frac{m_1}{M} \vec{r} \quad (t) = \vec{R}(t) + \vec{r}_{2\text{cm}}(t)
\end{align*}
\]

we can also relate the energy and angular momentum in the cm frame to their values in a moving frame

\[
\begin{align*}
\frac{1}{2} m_1 \dot{\vec{r}}^{\text{cm}}_1 + \frac{1}{2} m_2 \dot{\vec{r}}^{\text{cm}}_2 &= \\
\frac{1}{2} m_1 \frac{m_2}{M^2} \vec{R}^2 + \frac{1}{2} m_2 \frac{m_1}{M^2} \vec{R}^2 &= \\
\frac{1}{2} \frac{m_1 m_2}{M^2} (m_1 + m_2) \vec{R}^2 &= \frac{1}{2} M \dot{\vec{R}}^2
\end{align*}
\]

\[
T = \frac{1}{2} m_1 \dot{\vec{r}}^{\text{cm}}_1 + \frac{1}{2} m_2 \dot{\vec{r}}^{\text{cm}}_2 + \frac{1}{2} M \dot{\vec{R}}^2
\]

Similarly for the angular momentum

\[
\begin{align*}
m_1 \vec{r}_1\text{cm} \times \dot{\vec{r}}^{\text{cm}}_1 + m_2 \vec{r}_2\text{cm} \times \dot{\vec{r}}^{\text{cm}}_2 &= \\
\frac{m_1 m_2}{M^2} \vec{R} \times \dot{\vec{R}} + \frac{m_2 m_1}{M^2} \vec{R} \times \dot{\vec{R}} &= \\
\frac{m_1 m_2}{M} \vec{R} \times \dot{\vec{R}} &= M \vec{r} \times \dot{\vec{R}} = \vec{I} - M \vec{R} \times \dot{\vec{R}}
\end{align*}
\]

\[
\dot{\vec{I}} = m_1 \vec{r}_1\text{cm} \times \dot{\vec{r}}^{\text{cm}}_1 + m_2 \vec{r}_2\text{cm} \times \dot{\vec{r}}^{\text{cm}}_2 + M \vec{R} \times \dot{\vec{R}}
\]