Lecture 21

The 2-body problem

\[
\begin{align*}
    m_1 \ddot{\vec{r}_1} &= m_1 \vec{q} + \vec{F}_{12} \\
    m_2 \ddot{\vec{r}_2} &= m_2 \vec{q} + \vec{F}_{21}
\end{align*}
\]

\[
\text{2nd Law}
\]

where \( \vec{F}_{12} = -\vec{F}_{21} \)

\[
\text{3rd Law}
\]

adding

\[
\begin{align*}
    m_1 \ddot{\vec{r}_1} + m_2 \ddot{\vec{r}_2} &= (m_1 + m_2) \vec{q} \\
    \frac{d^2}{dt^2} \left( \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \right) &= \vec{q}
\end{align*}
\]

define

\[
\begin{align*}
    M &= m_1 + m_2 = \text{total mass of system} \\
    \vec{R} &= \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} = \text{center of mass coordinate}
\end{align*}
\]

It follows that

\[
\ddot{\vec{R}} = \vec{q}
\]

The system behaves like a free particle at \( \vec{R} \) with mass \( M \) in a uniform gravitational field.
It is also useful to define the relative coordinate

\[ \vec{F} = \vec{r}_1 - \vec{r}_2 \]

we can solve for \( \vec{r}_1, \vec{r}_2 \) in terms of \( \vec{F} \) and \( \vec{R} \)

\[
\begin{align*}
M\vec{R} &= m_1\vec{r}_1 + m_2\vec{r}_2 \\
 m_1\vec{F} &= m_1\vec{r}_1 - m_1\vec{r}_2 \\
m_2\vec{F} &= m_2\vec{r}_1 - m_2\vec{r}_2
\end{align*}
\]

adding the first and third lines gives

\[ M\vec{R} + m_2\vec{F} = (m_1 + m_2)\vec{r}_1 = Mr_1 \]

\[ \boxed{\vec{r}_1 = \vec{R} + \frac{m_2}{M} \vec{F}} \]

subtracting the second line from the first:

\[ M\vec{R} - m_1\vec{F} = (m_1 + m_2)\vec{r}_2 = Mr_2 \]

\[ \boxed{\vec{r}_2 = \vec{R} - \frac{m_1}{M} \vec{F}} \]

\[ \vec{r}'' = \vec{r}_1'' - \vec{r}_2'' = \frac{1}{m_1}(m_1\vec{g} + \vec{F}_{12}) - \frac{1}{m_2}(m_2\vec{g} + F_{21}) \]

\[ = (\vec{g} - \vec{g}) + \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{F}_{12} \]
we define \( \mu \) to be reduced mass by

\[
\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}
\]

\[
\mu \ddot{r} = F_{12}
\]

\[
M \ddot{R} = M \ddot{q}
\]

This kind of separation is seen in the kinetic energy and angular momentum.

\[
T = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2
\]

\[
= \frac{1}{2} m_1 \left( \ddot{R} + \frac{m_2}{M} \dot{r}_1 \right)^2 + \frac{1}{2} m_2 \left( \ddot{R} - \frac{m_1}{M} \dot{r}_1 \right)^2
\]

\[
= \frac{1}{2} (m_1 + m_2) \ddot{R}^2
\]

\[
\left( \frac{1}{2} \frac{m_1 m_2}{M^2} + \frac{1}{2} \frac{m_2 m_1}{M^2} \right) \dot{r}_1^2 +
\]

\[
\left( \frac{m_1 m_2}{M^2} - \frac{m_2 m_1}{M^2} \right) \ddot{R} \cdot \dot{r}_1
\]

\[
= \frac{1}{2} M \dot{R}^2 + \frac{1}{2} m_1 m_2 \left( \frac{m_1 + m_2}{M^2} \right) \dot{r}_1^2
\]

\[
= \frac{1}{2} M \dot{R}^2 + \frac{1}{2} \mu \dot{r}_1^2
\]

(Note that there are no cross terms \( \ddot{R} \cdot \dot{r}_1 \).)
Angular momentum

\[ \overline{J} = m_1 \overrightarrow{R}_1 \times \dot{\overrightarrow{R}}_1 + m_2 \overrightarrow{R}_2 \times \dot{\overrightarrow{R}}_2 \]

\[ = m_1 \left( \overrightarrow{R} + \frac{m_2}{M} \overrightarrow{F} \right) \times \left( \dot{\overrightarrow{R}} + \frac{m_2}{M} \dot{\overrightarrow{F}} \right) + m_2 \left( \overrightarrow{R} - \frac{m_1}{M} \overrightarrow{F} \right) \times \left( \dot{\overrightarrow{R}} - \frac{m_1}{M} \dot{\overrightarrow{F}} \right) \]

\[ = \left( m_1 + m_2 \right) \overrightarrow{R} \times \dot{\overrightarrow{R}} + \left( \frac{m_1 m_2^3 + m_2 m_1^3}{M^2} \right) \overrightarrow{F} \times \dot{\overrightarrow{F}} + \frac{m_1 m_2}{M} \left( \overrightarrow{F} \times \dot{\overrightarrow{R}} - \overrightarrow{R} \times \dot{\overrightarrow{F}} + \dot{\overrightarrow{R}} \times \dot{\overrightarrow{F}} - \overrightarrow{R} \times \dot{\overrightarrow{F}} \right) \]

\[ = M \overrightarrow{R} \times \dot{\overrightarrow{R}} + m \overrightarrow{F} \times \dot{\overrightarrow{F}} \]

\[ \Rightarrow \]

\[ L = \frac{1}{2} M \dot{\overrightarrow{R}}^2 + \frac{1}{2} m \dot{\overrightarrow{F}}^2 - MgZ - V(\overrightarrow{r}) \]

\[ T = \frac{1}{2} M \dot{\overrightarrow{R}}^2 + \frac{1}{2} m \dot{\overrightarrow{F}}^2 \]

\[ \overline{J} = M \overrightarrow{R} \times \dot{\overrightarrow{R}} + m \overrightarrow{F} \times \dot{\overrightarrow{F}} \]

It is useful to work in a coordinate system where \( \overrightarrow{R} = 0 \) (with a uniform gravitational field this is an accelerated coordinate system)
In this coordinate system
\[ \vec{r}_1 = \frac{m_1}{M} \vec{r} \]
\[ \vec{r}_2 = -\frac{m_1}{M} \vec{r} \]

The kinetic energy in the center of mass coordinate system is
\[
T_{cm} = \frac{1}{2} m_1 \dot{\vec{r}}_1^2 + \frac{1}{2} m_2 \dot{\vec{r}}_2^2
= \frac{1}{2} m_1 \left( \frac{m_1}{M} \dot{\vec{r}} \right)^2 + \frac{1}{2} m_2 \left( -\frac{m_1}{M} \dot{\vec{r}} \right)^2
= \frac{1}{2} \left( \frac{m_1 m_2^2}{M^3} + \frac{m_2 m_1^2}{M^2} \right) \dot{\vec{r}}^2
= \frac{1}{2} \frac{m_1 m_2}{M} \frac{m_2 + m_1}{M} \dot{\vec{r}}^2
= \frac{1}{2} M \dot{\vec{r}}^2
\]

The angular momentum in the center of mass is
\[
\vec{J}_{cm} = m_1 \vec{r}_1 \times \vec{\dot{r}}_1 + m_2 \vec{r}_2 \times \vec{\dot{r}}_2
= m_1 \frac{m_2}{M} \vec{r} \times \dot{\vec{r}} + m_1 \frac{m_1}{M} \vec{r} \times \dot{\vec{r}}
= m_1 m_2 \frac{m_2 + m_1}{M} \vec{r} \times \dot{\vec{r}}
= M \vec{r} \times \dot{\vec{r}}
This shows that

\[ T = T_{cm} + \frac{1}{2} M \dot{R} \]
\[ J = J_{cm} + M R^2 \times \dot{R} \]

This means that the total kinetic energy is the kinetic energy in the center of mass system plus the kinetic energy of a particle with the mass of the system moving with the velocity of the center of mass.

Similar remarks apply to the angular momentum—it is the sum of the angular momentum in the center of mass system plus the angular momentum of a particle with the mass of the system located at \( R \).
Once a problem has been solved in the center of mass system then

\[ \vec{P}_1 = \vec{R} + \frac{m_2 \vec{F}}{M} = \vec{R} + \vec{P}_{1\text{cm}} \]
\[ \vec{P}_2 = \vec{R} - \frac{m_1 \vec{F}}{M} = \vec{R} + \vec{P}_{2\text{cm}} \]
\[ \vec{r}_1 = \vec{R} + \vec{r}_{1\text{cm}} \]
\[ \vec{r}_2 = \vec{R} + \vec{r}_{2\text{cm}} \]
\[ \vec{P}_1 = m_1 \vec{\dot{R}} + \vec{P}_{1\text{cm}} \]
\[ \vec{P}_2 = m_2 \vec{\dot{R}} + \vec{P}_{2\text{cm}} \]
\[ \Gamma = \frac{1}{2} M \vec{\dot{R}}^2 + \tau_{\text{em}} \]
\[ \vec{J} = M \vec{R} \times \vec{\dot{R}} + \vec{\omega}_{\text{cm}} \]

Planets orbiting the sun

\[ M \vec{\ddot{r}} = -\frac{GMm_2}{r^3} \vec{r} = -GM \frac{m_1 m_2}{M} \frac{\vec{F}}{r^3} \]
\[ = -GM \frac{\vec{F}}{r^3} \]

In this case the \( M \) cancels and we get the same equation we previously obtained except

\[ M = m_s + m_p \]

So instead of \( M \) being the mass of the sun, it is the sum of the masses.
In the center of mass frame

\[ m_2 \vec{r}_{scw} = -m_1 \vec{r}_{scw} \]

This means that both particles orbit the center of mass which remains fixed.

Elastic collisions:

These are collisions that conserve momentum and kinetic energy.

We begin by considering the collision in the center of mass frame. In this case \( \vec{R} = 0 \):

\[ \vec{r}_{1\text{cm}} = \frac{m_2}{M} \vec{r} \quad m_1 \vec{r}_{1\text{cm}} + m_2 \vec{r}_{2\text{cm}} = 0 \]

\[ \vec{r}_{2\text{cm}} = -\frac{m_1}{M} \vec{r} \quad \vec{P}_{1\text{cm}} + \vec{P}_{2\text{cm}} = 0 \]

Let's denote the initial cm momenta by \( \vec{P}_{1\text{cm}} \) and \( \vec{P}_{2\text{cm}} \) and the final cm momenta by \( \vec{Q}_{1\text{cm}} \) and \( \vec{Q}_{2\text{cm}} \).
\\text{Therefore}
\\begin{align*}
\bar{P}_{1\text{cm}} &= -\bar{P}_{2\text{cm}} \\
\bar{q}_{1\text{cm}} &= -\bar{q}_{2\text{cm}}
\end{align*}
\\begin{align*}
T_x &= \frac{1}{2} m_1 \bar{P}_{1\text{cm}}^2 + \frac{1}{2} m_2 \bar{P}_{2\text{cm}}^2 = \\
&= \frac{1}{2} M \bar{P}_{\text{cm}}^2 \\
&= \frac{1}{2} m_1 \bar{q}_{1\text{cm}}^2 + \frac{1}{2} m_2 \bar{q}_{2\text{cm}}^2 \\
&= \frac{1}{2} M \bar{q}_{\text{cm}}^2
\end{align*}
\\text{Since } T_{\text{cm}} \text{ is conserved}
\\begin{align*}
|\bar{P}_{\text{cm}}|^2 &= |\bar{P}_{1\text{cm}}|^2 + |\bar{q}_{1\text{cm}}|^2 + |\bar{P}_{2\text{cm}}|^2 + |\bar{q}_{2\text{cm}}|^2
\end{align*}

This is all that can be learned from the conservation laws.

In many scattering experiments, one particle is initially at rest. The relevant coordinate system is called the laboratory coordinate system.

This can be treated by converting the cm results to the lab results.
to treat elastic collisions in the laboratory frame assume particle 2 is at rest

\[ \vec{q}_1 - \vec{q}_2 \]

momentum and energy conservation give

\[ \vec{p}_1 = \vec{q}_1 + \vec{q}_2, \]

\[ \frac{\vec{p}_1^2}{2m_1} = \frac{q_1^2}{2m_1} + \frac{q_2^2}{2m_2} \]

we analyze this system by relating laboratory quantities to cu quantities

\[ \vec{r}_{1\text{lab}} = \vec{r} + \vec{r}_{1\text{cm}} \quad \vec{r}_{1\text{cm}} = \frac{m_2}{M} \vec{r} \]

\[ \vec{r}_{2\text{lab}} = \vec{r} + \vec{r}_{2\text{cm}} \]

multiply by \( m_1, (m_2) \) and differentiating

\[ \vec{p}_{1L} = m_1 \dot{\vec{r}} + \vec{p}_{1\text{cm}} \]

\[ \vec{p}_{2L} = 0 = m_2 \dot{\vec{r}} + \vec{p}_{2\text{cm}} \]

the second equation gives

\[ \vec{p}_{1\text{cm}} = -m_2 \dot{\vec{r}} = -\vec{p}_{1\text{cm}} \]

\[ \dot{\vec{r}} = \frac{1}{m_2} \vec{p}_{1\text{cm}} \]

\[ m_1 \dot{\vec{r}} = \frac{m_1}{m_2} \vec{p}_{1\text{cm}} \]

\[ \vec{p}_{1L} = (1 + \frac{m_1}{m_2}) \vec{p}_{1\text{cm}} = \frac{M}{m_2} \vec{p}_{1\text{cm}} \]
It is useful to express the relations graphically using

\[ \ddot{p}_{1L} = \ddot{q}_{1L} + \ddot{q}_{2L} \]

\[ \ddot{q}_{1L} = m_1 \ddot{R} + \ddot{\alpha}_{cm} \]

\[ \ddot{q}_{2L} = m_2 \ddot{R} + \ddot{\beta}_{cm} = m_2 \ddot{R} - \dot{q}_{1cm} \]

\[ P_{2L} = 0 = m_2 \ddot{R} + P_{2cm} \]

\[ P_{1cm} = -P_{2cm} = m_2 \ddot{R} \]

\[ \ddot{\alpha}_{1L} = P_{1cm} + m_1 \ddot{R} = P_{1cm} + \frac{m_1}{m_2} \frac{\dot{P}_{1cm}}{\dot{R}} \]

\[ \cos \Theta = \dot{P}_{1L} \dot{q}_1 \]

\[ \cos \alpha = \dot{P}_{1L} \dot{q}_2 \]

\[ \cos \alpha_{cm} = \dot{P}_{1cm} \dot{q}_{cm} \]

Also note \( l \ddot{P}_{1cm} = l \ddot{q}_{cm} \)
\[ q_1 \sin \theta = P_{cm} \sin \theta_{cm} \]
\[ 2 \alpha = \pi - \theta_{cm} \]

**Using the Law of Cosines**

\[ a^2 + b^2 - c^2 = 2ab \cos \alpha \]
\[ P_{cm}^2 + P_{cm}^2 - q_2^2 = 2P_{cm}^2 \cos \theta \]
\[ q_2^2 = 2P_{cm}^2 (1 - \cos \theta_{cm}) \]
\[ q_2^2 = 4P_{cm}^2 \sin^2 (\frac{\theta}{2}) \]
\[ q_2 = 2P_{cm} \sin (\frac{\theta_{cm}}{2}) \]

\[ q_2^2 + p_t^2 - q_t^2 = 2q_2 p_t \cos \alpha \]
\[ q_t^2 = q_2^2 + p_t^2 - 2q_2 p_t \cos \alpha \]

These 4 equations give

\[ q_1, q_2, p_t \theta \]

in terms of \( p_t, P_{cm}, \theta_{cm} \)

\[ \tan \theta = \frac{P_{cm} \sin \theta_{cm}}{m_1 m_2 P_{cm} + P_{cm} \cos \theta_{cm}} = \frac{\sin \theta_{cm}}{\frac{m_1}{m_2} + \cos \theta_{cm}} \]
If \( m_1 > m_2 \) then \( m_1 \dot{R} > p_t \)

\[
1 \cdot \frac{m_1}{m_1} \cdot \frac{p_t}{m_1} \cdot \sin \theta = \frac{p_t'}{m_1} \cdot \frac{p_t'}{m_1} = \frac{m_1}{m_1}
\]

depending on the mass ratio — if a heavy particle scatters off of a light particle there is a maximal scattering angle.

The energy transferred to the target is

\[
\frac{q_t^2}{2m_2} = 4 \frac{p_{cm}^2 \sin^2 \left( \frac{\theta}{2} \right)}{2m_2} = 2 \frac{p_{cm}^2 \sin^2 \left( \frac{\theta}{2} \right)}{m_2}
\]

\[
\frac{p_t^2}{2m_1} = \left( 1 + \frac{m_1^2}{m_2^2} \right) \frac{p_{cm}^2}{2m_2^2} \frac{m_2^2}{2m_1} P_{cm}^2 = \frac{1}{2} \frac{M^2}{m_1} \frac{p_{cm}^2}{m_2}
\]

Taking the ratio gives

\[
\left( \frac{q_t^2}{2m_1} \right) \left( \frac{p_t^2}{2m_1} \right) = 4 \frac{8 \sin^2 \left( \frac{\theta}{2} \right) \mu}{1} \frac{M}{m_1}
\]

this is the fraction of energy transferred to the target.