Lecture 22

Last time

Center of mass - relative coordinates

\[ \vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M} \quad M = m_1 + m_2 \]

\[ \vec{r} = \vec{r}_1 - \vec{r}_2 \quad \mu = \frac{m m_1}{M} \]

Equations of motion

\[ M \ddot{\vec{R}} = M \vec{g} \quad \text{(external uniform grav. force)} \]

\[ \mu \ddot{\vec{r}} = \vec{F}(\vec{r}) \]

Inverting:

\[ \vec{r}_1 = \vec{R} + \frac{m_1}{M} \vec{r} \]

\[ \vec{r}_2 = \vec{R} - \frac{m_1}{M} \vec{r} \]

Nice feature

\[ T = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2 \]

\[ \dot{\vec{V}} = M \vec{R} \times \dot{\vec{R}} + \mu \vec{r} \times \dot{\vec{r}} \]

\[ L = \frac{1}{2} M \dot{\vec{R}}^2 - m g z + \frac{1}{2} \mu \dot{\vec{r}}^2 - V(\vec{r}) \]

One strategy is to first solve the problem in the CM system

(\( \vec{R} = \dot{\vec{R}} = 0 \))

\[ \vec{F}_{1 \text{CM}} = \frac{m_2}{M} \vec{F} \quad \vec{F}_{1 \text{CM}} = \frac{m_1 \vec{F}}{M} \]

\[ \vec{F}_{2 \text{CM}} = -\frac{m_1}{M} \vec{F} \quad \vec{F}_{2 \text{CM}} = -\frac{m_1 \vec{F}}{M} \]
\[ T_{cm} = \frac{1}{2} m_1 \dot{r}_{1cm}^2 + \frac{1}{2} m_2 \dot{r}_{2cm}^2 = \frac{1}{2} M \ddot{r}^2 \]

\[ \omega_{cm} = \frac{1}{2} m_1 \dot{r}_{1cm} \times \dot{r}_{1cm} + \frac{1}{2} m_2 \dot{r}_{2cm} \times \dot{r}_{2cm} = M \dot{\omega} \times \dot{r} \]

This means

\[ T = \frac{1}{2} M \ddot{r}^2 + T_{cm} \]

\[ \vec{J} = M \dot{r} \times \dot{r} + \omega_{cm} \]

Last time

orbits for inverse square

\[ \ddot{r} = -\frac{\mu M}{r^3} \vec{F} \]

\[ \dot{\vec{r}} = -\frac{\mu \vec{v}}{r^3} \vec{F} \quad (M = m_1 + m_2 \; \text{not} \; m_2) \]

(1) the cm does not move

(2) \( \vec{F} \) goes through the cm

(3) both particles move in this case

Elastic collisions

\[ \vec{p}_1, \vec{p}_2 \quad \text{momenta of each particle before collision} \]

\[ \vec{q}_1, \vec{q}_2 \quad \text{momenta of each particle after collision} \]
no external forces; forces derived from a potential

energy conservation
\[ \frac{\ddot{P}_1}{2m_1} + \frac{\ddot{P}_2}{2m_2} = \frac{\ddot{q}_1}{2m_1} + \frac{\ddot{q}_2}{2m_2} \]

momentum conservation
\[ \dot{P}_1 + \dot{P}_2 = \dot{q}_1 + \dot{q}_2 \]

The conservation laws have a simple form in the cu
\[ \ddot{R} = 0 \quad \dot{R} = 0 \quad \ddot{e} = 0 \]
\[ m_1 \ddot{r}_1 + m_2 \ddot{r}_2 = 0 \]
\[ \frac{\ddot{P}_1 + \ddot{P}_2}{2m} = \frac{\ddot{q}_1 + \ddot{q}_2}{2m} = 0 \]

\(\text{this means}\)
\[ \ddot{P}_{2cu} = -\ddot{P}_{1cu} \]
\[ \ddot{q}_{2cu} = -\ddot{q}_{1cu} \]

energy conservation in the cu gives
\[ \frac{1}{2} m_1 \dddot{P}_{1cu} + \frac{1}{2} m_1 \dddot{P}_{2cu} = \frac{1}{2} m_1 \dddot{q}_{1cu} + \frac{1}{2} m \dddot{q}_{2cu} \]
\[ \frac{1}{2} M \dddot{P}_{1cu} = \frac{1}{2} M \dddot{q}_{1cu} \]
So in the center of mass system
\[ |\vec{P}_{1cm}|^2 = |\vec{P}_{2cm}|^2 = |\vec{\alpha}_{1cm}|^2 = |\vec{\alpha}_{2cm}|^2 \]
\[ \vec{P}_{1cm} = -\vec{P}_{2cm} \quad \vec{\alpha}_{1cm} = -\vec{\alpha}_{2cm} \]

\[ \vec{P}_{1cm} \quad \Theta_{cm} \quad \vec{P}_{2cm} \]
\[ \vec{\alpha}_{2cm} \]

\( \Theta_{cm} \) is the center of mass scattering angle = deflection of particle 1 as a result of the collision.

These results can be used to treat the collision as a different coordinate system.

The most common frame is the laboratory frame where one particle is initially at rest.
To treat the collision in the laboratory coordinate system let $\vec{p}_2 = 0$. Then

\[
\begin{align*}
\vec{p}_{1L} &= m_1 \vec{R} + \vec{p}_{1cm} \\
\vec{p}_{2L} &= m_2 \vec{R} + \vec{p}_{2cm} = 0 \\
\vec{q}_{1L} &= m_1 \vec{\dot{R}} + \vec{q}_{1cm} \\
\vec{q}_{2L} &= m_2 \vec{\dot{R}} + \vec{q}_{2cm}
\end{align*}
\]

The $\vec{p}_1$ equation gives

\[\vec{p}_{2cm} = -m_2 \vec{\dot{R}} = -\vec{p}_{1cm}\text{ or } m_2 \vec{\dot{R}} = \vec{p}_{1cm}\]

To proceed it is useful to draw a graph showing

1. $\vec{q}_{1L} + \vec{q}_{2L} = \vec{p}_{1L}$
2. $\vec{p}_{1L} = m_1 \vec{\dot{R}} + \vec{p}_{1cm} = \frac{m_1}{m_2} \vec{p}_{2cm} + \vec{p}_{1cm}$
3. $\vec{q}_{1L} = m_1 \vec{\dot{R}} + \vec{q}_{1cm} = \frac{m_1}{m_2} \vec{p}_{2cm} + \vec{q}_{1cm}$
4. $\vec{q}_{2L} = m_2 \vec{\dot{R}} + \vec{q}_{2cm} = \vec{p}_{1cm} + \vec{q}_{2cm}$
given \( \vec{p}_i \), we would like to find all of the other relevant quantities:

1. \( |\vec{p}_{\text{cm}}| \)
2. \( \alpha \)
3. \( \Theta \)
4. \( q_1 \)
5. \( q_2 \)

\[ \vec{p}_{\text{cm}} = (1 + \frac{m_1}{m_2}) \vec{p}_i \]

\[ 2\alpha + \Theta_{\text{cm}} = \Pi \]

\[ \alpha = \frac{\Pi - \Theta_{\text{cm}}}{2} \]

\[ \tan \Theta = \frac{|P_{\text{cm}}| \sin \Theta_{\text{cm}}}{\frac{m_1}{m_2} |P_{\text{cm}}| + |P_{\text{cm}}| \cos \Theta_{\text{cm}}} \]

\[ \tan \Theta = \frac{\sin \Theta_{\text{cm}}}{\frac{m_1}{m_2} + \cos \Theta_{\text{cm}}} \]

4. \( q_1 \sin \Theta = |P_{\text{cm}}| \sin \Theta_{\text{cm}} \)

5. \[ |P_{\text{cm}}|^2 + |P_{\text{cm}}|^2 - q_2^2 = 2 |P_{\text{cm}}|^2 \cos \Theta_{\text{cm}} \]

\[ q_2^2 = \frac{2 |P_{\text{cm}}|^2 (1 - \cos \Theta_{\text{cm}})}{4 P_{\text{cm}}^2 \sin^2 \left( \frac{\Theta_{\text{cm}}}{2} \right)} \]

This shows that all interesting quantities in the lab frame can be determined from \( p_i \) and the cm scattering angle.
energy transferred to particle 2

\[ T_1 = \frac{p_1^2}{2m_1} = \frac{1}{2m_1} \left( 1 + \frac{m_1}{m_2} \right)^2 p_{cm}^2 \]

\[ T_2' = \frac{q^2}{2m_2} = \frac{4 p_{cm}^2 \sin^2 \left( \frac{\Theta_{cm}}{2} \right)}{2m_2} \]

\[ \frac{T_2'}{T_1} = \frac{4 p_{cm}^2 \sin^2 \left( \frac{\Theta_{cm}}{2} \right)}{2m_1} \frac{2m_1}{\left( 1 + \frac{m_1}{m_2} \right)^2 p_{cm}^2} \]

\[ \frac{T_2'}{T_1} = 4 \frac{m_1}{m_2} \frac{\sin^2 \left( \frac{\Theta_{cm}}{2} \right)}{\left( 1 + \frac{m_1}{m_2} \right)^2} \]

\[ = 4 \frac{\sin^2 \left( \frac{\Theta_{cm}}{2} \right)}{\frac{m_2}{m_1} \frac{m_2}{m_1}} \]

\[ = 4 \frac{\sin^2 \left( \frac{\Theta_{cm}}{2} \right)}{\frac{m_2}{m_1}} \]

This is maximized when \( \Theta_{cm} = \pi \)

This corresponds to a head-on collision.

Remark: When \( m_1 > m_2 \) then

\[ \frac{m_1}{m_2} |p_{cm}| > |p_{cm}| \]

\[ \sin \Theta_{cm} = \frac{|p_{cm}|}{\frac{m_1}{m_2} |p_{cm}|} = \frac{m_2}{m_1} \text{ largest scattering angle in the lab} \]
Two-particle scattering

Laboratory scattering

1. \( \tilde{s}_I = \frac{\text{flux of incoming particle}}{(\text{unit area})(\text{unit time})} \)

2. \# scattered per unit time

\[
\omega_I = \tilde{s}_I \sigma \cdot n_2 V
\]

\[
d\omega = \int \frac{dA}{l^2} n_1 V d\Omega
\]

\# scattered / time in solid angle \( d\Omega \)

Center of mass scattering

\[
\omega = n_1 v_{12} n_2 V \sigma
\]

\[
d\omega = n_1 n_2 v_{12} V \frac{d\sigma}{d\Omega} d\Omega
\]

Consider the case of hard sphere scattering in the center of mass.
In this case

1. \[ \Theta = 2\beta + \Theta_{cm} \]
   \[ \beta = \frac{\Theta - \Theta_{cm}}{2} \]

2. \[ b = (a_1 + a_2) \sin \beta \]
   \[ = (a_1 + a_2) \sin \left( \frac{\Theta - \Theta_{cm}}{2} \right) \]
   \[ = (a_1 + a_2) \left( \sin \left( \frac{\Theta}{2} \right) \cos \left( \frac{\Theta_{cm}}{2} \right) - \sin \left( \frac{\Theta_{cm}}{2} \right) \cos \left( \frac{\Theta}{2} \right) \right) \]
   \[ = (a_1 + a_2) \cos \left( \frac{\Theta_{cm}}{2} \right) \]

The differential cross section is

\[ d\sigma = bdbd\phi = (a_1 + a_2)^2 \cos \left( \frac{\Theta_{cm}}{2} \right) \cdot \left( - \frac{1}{2} \sin \left( \frac{\Theta_{cm}}{2} \right) \right) d\Theta_{cm} d\phi \]
\[ = \frac{1}{4} (a_1 + a_2)^2 \sin \left( \Theta_{cm} \right) d\Theta_{cm} d\phi_{cm} \]

\[ \left( \frac{d\sigma}{d\Omega} \right)_{cm} = \frac{1}{4} (a_1 + a_2)^2 \]
\[ \sigma_{cm} = \int d\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{1}{4} (a_1 + a_2)^2 4\pi = \pi (a_1 + a_2)^2 \]

which is the expected result. If we transform the system to the lab frame, the total cross section does not change - it is just the area seen head on.

To calculate the differential cross section in the lab or cm coordinate system, we note

\[ \int \left( \frac{d\sigma}{d\Omega} \right)_{cm} \sin \theta_{cm} d\theta_{cm} d\phi_{cm} = \]

\[ \int \left( \frac{d\sigma}{d\Omega} \right)_{L} \sin \theta_{L} d\theta_L d\phi_L \]

\[ \phi_L = \Phi_{cm} \]

\[ \int \left( \frac{d\sigma}{d\Omega} \right)_{L} \sin \theta_L \frac{d\theta_L}{d\theta_{cm}} \sin \theta_{cm} \]

Comparing these two expressions:

\[ \left( \frac{d\sigma}{d\Omega} \right)_{cm} = \left( \frac{d\sigma}{d\Omega} \right)_{L} \frac{d\theta_L}{d\theta_{cm}} \]

or

\[ \left( \frac{d\sigma}{d\Omega} \right)_{L} = \left( \frac{d\sigma}{d\Omega} \right)_{cm} \frac{d\theta_{cm}}{d\theta_L} \]
mis can be computed using the relation

$$\tan \theta = \frac{\sin \theta_{cm}}{\frac{m_1}{m_2} \cos \theta_{cm}}$$

by squaring, this we can relate the cosines:

$$\frac{1 - \cos^2 \theta_{cm}}{\cos \theta_{cm}} = \frac{1}{\cos \theta_{cm}} - 1 = \frac{1 - \cos \theta_{cm}}{\left(\frac{m_1}{m_2}\right)^2 + 2\left(\frac{m_1}{m_2}\right) \cos \theta_{cm} + \cos^2 \theta_{cm}}$$

Let $Z = \cos \theta_{cm}$, $Z_{cm} = \cos \theta_{cm} = 1$,

$$\frac{1}{Z^2} = 1 + \frac{1 - Z_{cm}^2}{\left(\frac{m_1}{m_2}\right)^2 + 2\left(\frac{m_1}{m_2}\right) Z_{cm} + Z_{cm}^2} = \frac{\left(\frac{m_1}{m_2}\right)^2 + 2\left(\frac{m_1}{m_2}\right) Z_{cm} + 1}{\left(\frac{m_1}{m_2} + Z_{cm}\right)^2}$$

$$Z^2 = \frac{\left(\frac{m_1}{m_2} + Z_{cm}\right)^2}{1 + 2\frac{m_1}{m_2} Z_{cm} + \left(\frac{m_1}{m_2}\right)^2}$$

$$dZ = -\sin \theta_{cm} d\theta_{cm}, \quad dZ_{cm} = -\sin \theta_{cm} d\theta_{cm}$$

Consider the case $m_1 = m_2$ then

$$Z^2 = \frac{\left(1 + Z_{cm}\right)^2}{2(1 + Z_{cm})} = \frac{1}{2} \left(1 + Z_{cm}\right)$$
Note this means
\[
\cos^2 \Theta = \frac{1 + \cos \Theta_{cm}}{2} = \cos^2 \left( \frac{\Theta_{cm}}{2} \right)
\]
\[
\Theta_L = \frac{\Theta_{cm}}{2}
\]
we also have
\[
2 \pi d \Omega = \frac{1}{2} d \Omega_{cm}
\]
the differential cross sections are related by
\[
\left( \frac{d \sigma}{d \Omega} \right)_L = \left( \frac{d \sigma}{d \Omega} \right)_{cm} d \Omega_{cm}
\]
\[
\left( \frac{d \sigma}{d \Omega} \right)_L = \left( \frac{d \sigma}{d \Omega} \right)_{cm} \frac{d \Omega_{cm}}{d \Omega} = 4 \pi \left( \frac{d \sigma}{d \Omega} \right)_{cm}
\]
\[
= 4 \cos \Theta_L \left( \frac{d \sigma}{d \Omega} \right)_{cm}
\]
we can check that this gives the same total cross section in the hard sphere
\[
\int \frac{d \sigma}{d \Omega} \sin \Theta_L \, d \Theta_L \, d \phi_L
\]
\[
( \Theta_L = \frac{\Theta_{cm}}{2} : 0 \rightarrow \frac{\pi}{2} )
\]
\[
\int \frac{d \sigma}{d \Omega} \frac{4}{3} \sin \Theta_L \cos \Theta_L \, d \Theta_L \, d \phi_L =
\]
for the hard sphere \( \left( \frac{d \sigma}{d \Omega} \right)_{cm} = \frac{a_i + a_j}{4} \)
\[
\int_0^{2\pi} \int_0^{\pi} d\phi_L \left( \frac{a_i + a_j}{4} \right) 2 \sin 2\Theta_L =
\]
\[
\frac{2\pi}{a_i + a_j} (-2)^{\frac{1}{2}} \cos(2\Theta) \bigg|_0^{\pi} \frac{2\pi}{4} (a_i + a_j) (-1)(-1) = \pi(a_i + a_j)
\]
which is the same cross section obtained in the cm frame

Summary:

using cm + relative coordinates split the problem into
motion of the center of mass and motion about the center
of mass. The relative motion
looks like single particle motion
with $M - u = \frac{m_1}{m_1 + m_2} F_1 - F = \vec{F}_1 - \vec{F}_2$

It is useful to first solve the problem
in the center of mass and
then translate coordinates.

Many Body System

Many of the concepts used in the
2 body system generalize to the
many particle system
systems of many particles

\[ m_n \frac{d^2 \vec{r}_n}{dt^2} = 2 \sum_{k \neq n} F_{nk} + \vec{F}_n \]

where \( F_{nk} \) is the force on the \( n \)th particle due to the \( k \)th particle and \( \vec{F}_n \) is an external force.

As before we define

\[ M = \sum_{n=1}^{N} m_n \text{ total mass of system} \]

\[ \overline{R} = \frac{1}{M} \sum_{n=1}^{N} m_n \vec{r}_n \text{ center of mass of the system} \]

Consider

\[ \sum_{n} m_n \frac{d^2 \vec{r}_n}{dt^2} = 2 \sum_{n \neq k} F_{nk} + 2 \vec{F}_n \]

\[ = \frac{1}{2} \sum_{n \neq k} (F_{nk} + F_{kn}) + \sum_{n} \vec{F}_n \]

vanish by sum of external forces third law hence \( \vec{F}_n \)

we can write this as

\[ M \frac{d^2}{dt^2} \left( \frac{1}{M} \sum_{n} m_n \vec{r}_n \right) = \frac{1}{2} \sum_{n} \vec{F}_n \]
or \[ \vec{M}\ddot{\vec{r}} = \vec{F}_{\text{ext}} \]

This looks just like Newton's second law for a particle in a force except:

- particle \rightarrow \text{system}
- coordinate \rightarrow \text{coordinate of CM}
- \( m_0 \rightarrow 2M \) total mass
- \( \vec{F} \rightarrow 2\vec{F}_n \) external force.

This justifies treating a marble as a point particle - it shows that much of what we did in a single particle also applies to systems of particles.

In general, the motion of individual particles in the system could be very complicated.

If \( \vec{F}_{\text{ext}} = 0 \) then \( \vec{M}\ddot{\vec{r}} = \vec{P} = \sum m_i \vec{v}_i \)

is conserved, this is conservation of linear momentum.
In systems it is useful to work in the center of mass system (i.e., where \( \overline{\mathbf{R}} = \dot{\overline{\mathbf{R}}} = 0 \))

we define

\[
\overline{\mathbf{F}}_n = \overline{\mathbf{F}} + \overline{\mathbf{F}}_{\text{cm}}
\]

where

\[ \sum m_i \overline{F}_{\text{cm}} = 0 \]

so the cm coordinates are not independent

example - rocket propulsion

Assume a rocket initially at rest emits dm mass of gas per unit time with velocity \( \mathbf{u} \) relative to the rocket.

Momentum conservation gives

\[
 m \mathbf{V} = (m-dm)(\mathbf{v}+d\mathbf{v}) + dm(\mathbf{v}-\mathbf{u}) = 0
\]

\[
 m \, dv = vdm + dm \, dv + dm(\mathbf{v}-\mathbf{u}) = 0
\]

\[
 m \frac{dv}{dm} + dv - \mathbf{v} = 0
\]

in the limit \( dm \to dv \to \)

\[
 \frac{dm}{dv} = \frac{m}{u} \quad m
\]
\[
\ln M = -\frac{v}{u} + \text{const}
\]
\[
M = C e^{-\frac{v}{u}}
\]
\[
M = M(0) e^{-\frac{v}{u}}
\]
\[
v = u \ln \frac{M(0)}{M}
\]