Lecture 23

Last time 2 body elastic collisions

\[ \bar{q}_1, \bar{q}_2, \bar{q}_{1cm}, \bar{q}_{2cm} \]

\[ \theta_L, \theta_{cm}, \alpha_L, \alpha_{cm} \]

\[ \frac{m_1}{m_2} \bar{P}_{cm}, \bar{P}_1, \bar{P}_{cm} \]

Relations

1. \( \bar{P}_{1L} = (1 + \frac{m_1}{m_2}) \bar{P}_{cm} \)
2. \( \alpha = \frac{\theta_{cm}}{2} \)
3. \( \tan \theta_L = \frac{\sin \theta_{cm}}{\frac{m_1}{m_2} + \cos \theta_{cm}} \)
4. \( q_{1L} \sin \theta_L = P_{cm} \sin \theta_{cm} \)
5. \( q_2^2 = 4 P_{cm}^2 \sin^2 \left( \frac{\theta_{cm}}{2} \right) \)

Strategy

1. Find \( \bar{P}_{cm} \)
2. Solve for \( \theta_{cm} \) in the cm frame
3. Express lab quantities in terms of cm quantities
Note that in the cm frame

\[ N_1 (v_1 - v_2) \cdot N_2 \propto v \]

\[ \text{# particle} \quad \text{# scattered time/area} \]

\[ dW = v_{rel} N_1 N_2 \sigma \]

\[ dW = v_{rel} N_1 N_2 \frac{d\sigma}{d\Omega} \, d\Omega \]

The differential cross section provides a measure of the relative probability that particles will be scattered into different solid angles.

Important: \( \sigma \) = total cross section is independent of the choice of coordinate system, \( \frac{d\sigma}{d\Omega} \) is not.
scattering LCG frame

\[ \frac{d\omega}{d\Omega} = \frac{\text{# particles scattered}}{\text{unit target volume}} \]

\[ = \int d\sigma \cdot n_t V / V = \int \sigma n_t \]

\[ \text{flux} \]

\[ f = \text{flux} = \frac{\text{# particles}}{\text{time} \cdot \text{area}} \]

\[ \sigma = \text{cross section} / \text{particle} \]

\[ n_t = \text{# target particles} / \text{volume} \]

\[ V = \text{volume}. \]

If we want \# scattered in \( d\Omega \) per unit time per unit volume

\[ \int \frac{d\omega}{d\Omega} \cdot n_t d\Omega. \]

\[ \frac{d\sigma}{d\Omega} \] is called the differential cross section

\[ d\Omega = \text{solid angle} = \sin \theta d\theta d\phi \]
How are they related

$$\int (\frac{d\phi}{d\tau})_L \sin \phi \, d\phi \, d\phi_L = \sigma =$$

$$\int (\frac{d\phi}{d\tau})_m \sin \phi_m \, d\phi_m \, d\phi_m$$

we have $\phi_m = \phi_L$ but

$$\sin \phi_L \, d\phi_L \neq \sin \phi_m \, d\phi_m$$

comparing the above expressions

$$\frac{d\phi}{d\tau}_L = \frac{d\phi}{d\tau}_m \frac{\sin \phi_m \, d\phi_m}{\sin \phi_L \, d\phi_L}$$

It is useful to use

$$Z_L = \cos \phi_L \quad dZ_L = -\sin \phi_L \, d\phi_L$$

$$Z_m = \cos \phi_m \quad dZ_m = -\sin \phi_m \, d\phi_m$$

$$\frac{\sin \phi_m \, d\phi_m}{\sin \phi_L \, d\phi_L} = \frac{dZ_m}{dZ_L}$$

$$\Rightarrow \quad (\frac{d\phi}{d\tau}_L) = (\frac{d\phi}{d\tau}_m) \frac{dZ_m}{dZ_L}$$

In order to find $Z_m (Z_L)$ we use

$$\tan \Theta_L = \frac{\sin \Theta_m}{\frac{m_1}{m_2} + \cos \Theta_m}$$
It is useful to express everything in terms of \( \cos \theta_c \) and \( \cos \theta_{cm} \)

\[
\tan^2 \theta_L = \frac{\sin^2 \theta_c}{\cos^2 \theta_c} = \frac{1 - \cos^2 \theta_c}{\cos^2 \theta_c} = \frac{1}{\cos^2 \theta_c} - 1
\]

\[
\sin^2 \theta_{cm} \left( \frac{m_1}{m_2} + \cos \theta_{cm} \right)^2 = \frac{1 - \cos^2 \theta_{cm}}{\left( \frac{m_1}{m_2} + \cos \theta_{cm} \right)^2}
\]

\[
\frac{1}{\cos \theta_L^2} = 1 + \frac{1 - \cos^2 \theta_{cm}}{\left( \frac{m_1}{m_2} + \cos \theta_{cm} \right)^2}
\]

\[
Z_L^2 = \frac{\left( \frac{m_1}{m_2} + Z_{cm} \right)^2}{\left( \frac{m_1}{m_2} \right)^2 + 2 \frac{m_1}{m_2} Z_{cm} + 1}
\]

This expression can be used to calculate

\[
\frac{dZ_{cm}}{dZ_L} = \left( \frac{dZ_{cm}}{dZ_{cm}} \right)^{-1}
\]

Some comments about scattering:

\[
\sin \theta_L = \frac{P_{cm}}{m_1 P_{cm}} = \frac{m_2}{m_1} \theta_c
\]

\[
\theta_{cm} = \frac{m_1}{m_2} \theta_{cm}
\]

\[
\frac{m_1}{m_2} P_{cm}
\]
This means that when the incoming particle is more massive than the target particle there is a maximum scattering angle in the laboratory frame.

We can also ask how much energy is transferred to the target particle.

* Initial energy

\[
\frac{p^2_{1c}}{2m_1} = \frac{E}{2m_1} = (1 + \frac{m_1}{m_2})^2 \frac{p^2_{cm}}{2m_1}
\]

* Final energy of target particle

\[
\frac{q^2_{2c}}{2m_2} = 4 \frac{p^2_{cm} \sin^2 \left( \frac{\Theta_{cm}}{2} \right)}{2m_2} = 2 \frac{p^2_{cm}}{m_2} \sin^2 \left( \frac{\Theta_{cm}}{2} \right)
\]

The ratio is:

\[
\frac{E_{2 \text{ final}}}{E_{\text{ initial}}} = \frac{2 \frac{p^2_{cm}}{m_2} \sin^2 \left( \frac{\Theta_{cm}}{2} \right)}{2m_1 \frac{p^2_{cm}}{m_2} (1 + \frac{m_1}{m_2})^2}
\]

\[
= 4 \frac{m_1}{m_2} \sin^2 \left( \frac{\Theta_{cm}}{2} \right) = 4 \frac{m_1 m_2}{m_2} \frac{\sin^2 \left( \frac{\Theta_{cm}}{2} \right)}{m_2}
\]

\[
= 4 \frac{m_1}{m_2} \sin^2 \left( \frac{\Theta_{cm}}{2} \right)
\]

The largest energy transfer occurs when \( \Theta_{cm} = \pi \) — corresponding to a head-on collision.
Consider the example of hard sphere scattering for identical spheres.

\[ G = \pi (r + r)^2 = 4\pi r^2 \]

\[ \frac{dG}{d\Omega}_{\text{cm}} = \frac{1}{4} (r + r)^2 = r^2 \]

To calculate the laboratory cross section

\[ \Theta_{\text{cm}} = 2\Theta_L \]

From geometry, \( \frac{m_1}{m_2} P_{\text{cm}} = P_{\text{cm}} = 1\)

\[ \Theta_{\text{cm}} = 2\Theta_L \]

\[ \frac{dG}{d\Omega}_L = \left( \frac{dG}{d\Omega}_{\text{cm}} \right)_{\text{cm}} \frac{dz_{\text{cm}}}{dz_L} \]

\[ z_L^2 = \frac{(1 + z_{\text{cm}})^2}{2(1 + z_m)} = \frac{1 + z_{\text{cm}}}{2} \]

\[ z_{\text{cm}} = 2z_L - 1 \]

\[ \frac{dz_{\text{cm}}}{dz_L} = 4z_L = 4 \cos \theta_L \]
This means that for identical hard spheres

\[
\left( \frac{d\sigma}{d\Omega_1} \right)_L = \left( \frac{d\sigma}{d\Omega_1} \right)_m \cdot 4 \cos \theta_L
\]

\[= 4 r^2 \cos \theta_L \]

which shows that the laboratory differential cross section is angle dependent.

\[
\sigma = \int_0^{2\pi} \sin \theta_L \, d\omega_L \int_0^{\pi/2} \, d\theta_L \cdot 4 r^2 \cos \theta_L =
\]

\[
\left( \text{this goes to } \frac{\pi}{2} \text{ because } \theta_L = \frac{\theta_{cm}}{2} \right)
\]

\[= 2\pi \cdot 4 r^2 \int_0^{\pi/2} \sin \theta_L \cos \theta_L \, d\omega_L \]

\[= 8\pi r^2 \int_0^{\pi/2} \sin (2\theta_L) \, d\omega_L \]

\[= 4\pi r^2 \cdot \frac{1}{2} \left( \cos (2\theta_L) \right) \bigg|_0^{\pi/2} \]

\[= 4\pi r^2 \cdot \frac{1}{2} (-1) (-1-1) \]

\[= 4\pi r^2 \]

which is identical to the cm cross section.
many body problem

\[ m_n \frac{d^2 \vec{r}_n}{dt^2} = \sum_{m \neq n} \vec{F}_{nm} + \vec{F}_n \quad M = \sum m_n \]

If we add these equations:

\[
\frac{d^2}{dt^2} \left( \frac{1}{n} \sum m_n \vec{r}_n \right) = \sum_{m \neq n} \vec{F}_{nm} + \vec{F}_n
\]

\[
= \sum_{m \neq n} \left( \vec{F}_{nm} + \vec{F}_{mn} \right) + \vec{F}_n
\]

(by third law)

\[
M \frac{d^2}{dt^2} \left( \frac{1}{M} \sum m_n \vec{r}_n \right) = \frac{2}{n} \vec{F}_n
\]

we define the center of mass coordinate

\[
\vec{R} = \frac{1}{M} \sum m_n \vec{r}_n
\]

\[
M \frac{d^2 \vec{R}}{dt^2} = \vec{F}_{ext} \quad (\vec{F}_{ext} = \sum \vec{F}_n)
\]

This is an important equation -
consider the motion of a baseball.
It is a complex many-body system -
but this equation means that it
can be treated as a point particle.
acted on by external force

If the system is isolated then there are no external forces. It follows that

$$\sum \dot{\textbf{r}}_i = \sum m_i \dot{\textbf{v}}_i = \text{constant}$$

This means that momentum is conserved

$$\sum m_i \dot{\textbf{v}}_i = \text{constant}$$

Application: Rocket propulsion

Consider a rocket of mass $M_0$ that emits $dm$ of gas at velocity $u$ relative to the rocket.

Consider the rocket and gas as an isolated system.

Initial momentum $Mv$

Final momentum $(M - dm)(V + dv) + dm(V - u)$

Momentum conservation

$$M\dot{v} = M\dot{v}_f + Mdv - dm\dot{v} - dm dv + dm\dot{v} - dm u$$

$$Mdv - dm u - dm dv = 0$$
letting \( dM = -dm \)

\[ MdV = -dMU - dMDV \]

divide by \( dv \) setting \( dM dv \)

\[ M = -\frac{dM}{dv} u - dv = -\frac{dM}{dv} u \]

or \[ \frac{dM}{M} = -\frac{dv}{u} \]

integrating this

\[ \ln M = -\frac{V}{u} + C \]

\[ M = M_0 e^{-\frac{V}{u}} \]

where \( M_0 \) is the rest mass of the rocket

This shows that the rocket loses fuel exponentially - we get

\[ V = u \ln \frac{M_0}{M} \]

Angular momentum

\[ \overrightarrow{L} = \overrightarrow{p} \times \overrightarrow{r} \]
\[
\frac{dJ}{dt} = 2 \left( \sum_{n} \bar{r}_n \times \ddot{\bar{r}}_n + \sum_{n} \bar{r}_n \times \dot{\bar{r}}_n \right)
= 2 \bar{r}_n \times \left( \sum_{m=n}^{\infty} \vec{F}_{nm} + \bar{F}_n \right)
\]

Consider
\[
\bar{r}_n \times \vec{F}_{nm} + \bar{r}_n \times \vec{F}_{mn} =
(\bar{r}_n - \bar{r}_m) \times \vec{F}_{nm}
\]

If \( \vec{F}_{nm} \) is a central force then \( \vec{F}_{nm} \) is in the direction \( \bar{r}_n - \bar{r}_m \) and this cross product vanishes. What remains

\[
\frac{dJ}{dt} = 2 \bar{r}_n \times \bar{F}_n = 2 \text{ external torques (central force)}
\]

It is useful to do what we did in the two body case -

\[
\bar{r}_n = \bar{R} + \bar{r}_{ncm}
\]

\[
2 \sum_{n} m_n \bar{r}_n = 2 m_n \bar{R} + 2 \sum_{n} m_n \bar{r}_{ncm}
\]

\[
\underbrace{M\bar{R}} \quad \underbrace{M\bar{R}} \quad \underbrace{0}
\]
Thus
\[ \dot{J} = \sum g \, m_n \, \bar{r}_n \times \dot{\bar{r}}_n \]
\[ = \sum g \, m_n \, (\ddot{\bar{R}} + \dot{\bar{F}}_{ncm}) \times (\dot{\bar{R}} + \dot{\bar{F}}_{ncm}) \]
\[ = \sum g \, m_n \, \bar{R} \times \dot{\bar{R}} + \sum g \, m_n \, \bar{F}_{ncm} \times \dot{\bar{r}}_{ncm} + \]
\[ \bar{R} \times (\sum g \, m_n \, \dot{\bar{r}}_n) + (\sum g \, m_n \, \dot{\bar{r}}_n) \times \dot{\bar{R}} \]
\[ \frac{d}{dt} (\bar{C}) \]

So what remains is
\[ \dot{J} = M \bar{R} \times \ddot{\bar{R}} + \sum g \, m_n \, \bar{F}_{ncm} \times \dot{\bar{r}}_{ncm} \]

\[ \dot{J}_{cm} = \sum g \, m_n \, \bar{F}_{ncm} \times \dot{\bar{F}}_{ncm} \]

We can compute the
\[ \bar{J} = M \bar{R} \times \ddot{\bar{R}} + \dot{J}_{cm} \]
\[ = \sum g \, (\ddot{\bar{R}} + \dot{\bar{F}}_{cm}) \times \bar{F}_n \]
\[ \text{central forces} \]
\[ = M \bar{R} \times \ddot{\bar{R}} + \sum \bar{F}_{ncm} \times \bar{F}_n \]

Comparing these expressions for central forces
\[ \frac{d}{dt} (\bar{J}_{cm}) = \sum \bar{F}_{ncm} \times \bar{F}_n \]
example - earth moon system

If we ignore the effect of the sun in this system there are no external forces, and the gravitational force is central.

This means the angular momentum about the center of mass is conserved.

There are 3 contributions:

\[ \vec{J}_{cm} = \vec{J}_e + \vec{J}_m + \mu \vec{r} \times \vec{v} \]

where \( \vec{J}_e \) and \( \vec{J}_m \) are the angular momenta of the earth and moon about their own axes.

We assume

1. the orbit is circular
2. all 3 vectors point in the same direction
3. \( \vec{v} \) is small compared to the other 2 terms.

both of \( \vec{J}_e \) and \( \mu \vec{r} \times \vec{v} \) are approximately separately conserved
\( \bar{I} \) is proportional to its rotational angular velocity \( \bar{\omega} \)

\[ \bar{I} = I \bar{\omega} \]

while the relative angular momentum

\[ m \bar{F} \times \bar{r} = ma^2 \bar{\Omega} \]

where \( \bar{\Omega} \) is the angular frequency about the moon.

Since this is approximately conserved, we can apply Kepler's third law:

\[ \Omega^2 a^3 = GM \]

\[ \Omega = \sqrt{\frac{GM}{a^3}} \]

\[ I \bar{\omega} + m \sqrt{\frac{GM}{a^3}} = \text{constant} \]

While these quantities are approximately separately conserved, due to some dissipative effects \( \bar{\omega} \) is slowing down; in order to preserve the total angular momentum the earth-moon distance \( a \) must increase—this effect is observable.
Energy Conservation

\[ T = \frac{1}{2} \sum m_n \dot{R}_n^2 \]

like in the two body case using

\[ \ddot{r}_{nm} = \ddot{R} + \ddot{r}_{nm} \quad \sum m_n \ddot{r}_{nm} = 0 \]

the rate of change of \( T \) with time

\[ \dot{T} = \frac{1}{2} \sum m_n \left( \dot{R} + 2 \ddot{r}_n \right) \cdot \ddot{r}_n \]

\[ = \sum m_n \dot{r}_n \left( \dddot{r}_n + \dddot{r}_{nm} \right) \]

If the internal forces are conservative then

\[ \sum m_n \dddot{r}_{nm} = - \sum m_n \vec{V}_{int} \cdot \vec{r}_n \]

\[ \dot{T} = \sum m_n \dddot{r}_n \cdot \vec{F}_{ext} \]

\[ = \sum m_n \dddot{r}_n \cdot \vec{F}_{ext} \]

\[ \Rightarrow \quad \frac{d}{dt} \left( T + V_{int} \right) = \sum m_n \dddot{r}_n \cdot \vec{F}_{ext} \]

If the external forces are also conservative

\[ \vec{F}_{ext} = - \nabla V \cdot \hat{r} \]

\[ \sum m_n \dot{r}_n \cdot \vec{F}_{ext} = - \sum m_n \nabla V \cdot \dddot{r}_n \]

\[ = - \sum m_n \nabla V \cdot \frac{d \vec{r}_n}{dt} = - \frac{d V_{ext}}{dt} \]
This gives

\[ \frac{d}{dt} \left( T + V_{\text{int}} + V_{\text{ext}} \right) = 0 \]

so energy conservation holds if both the internal and external forces are conservative.

Lagrange's equations consider the case where all of the forces are conservative.

\[ m_n \ddot{x}_n = -\frac{\partial V}{\partial x_n} \]
\[ m_n \ddot{y}_n = -\frac{\partial V}{\partial y_n} \]
\[ m_n \ddot{z}_n = -\frac{\partial V}{\partial z_n} \]

\[ \left( \frac{d}{dt} \left( \frac{\partial (T)}{\partial \dot{x}_n} \right) - \frac{\partial (T)}{\partial x_n} \right) (T-V) = m_n \dddot{x}_n + \frac{\partial V}{\partial x_n} = 0 \]

Just by comparison we see Lagrange's equations are completely equivalent to Newton's laws.
These can be derived using the principle of stationary action.

A change vanishes

$$\bar{r}_n = \bar{r}_n(q_i - q_{i0} t)$$

$$T = T(q_i, q_{i0}, \dot{q}_i, \dot{q}_{i0})$$

$$V = V(q_i, q_{i0})$$

$$L = T - V$$

$$\delta \left[ \bar{r}_n \cdot t \cdot t \right] = \int_{t_1}^{t_2} L(q_i, q_{i0}, \dot{q}_i, \dot{q}_{i0}, t) \, dt$$

$$q_{i0}(t) = q_{i0}(t) + \lambda \delta q_i(t)$$

$$\delta q_{i0}(t_1) = \delta q_i(t_1) = 0$$

$$\frac{d}{dt} \left[ \bar{r}_n + \lambda \delta \dot{r}_n \right]_{t=t_1} = 0$$

$$\int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q_{i0}} \dot{q}_{i0} + \frac{\partial L}{\partial \dot{q}_{i0}} \ddot{q}_{i0} \right) \, dt =$$

$$\int_{t_1}^{t_2} - \left( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_{i0}} \right) + \frac{\partial L}{\partial q_{i0}} \right) \ddot{q}_{i0} \, dt + \int_{t_1}^{t_2} 2 \frac{\partial \delta L}{\partial \delta q_{i0}} \Big|_{t_1}^{t_2}$$

vanishes @ t, t_1

For this to vanish, let all $$\delta q_{i0} = 0$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_{i0}} \right) - \frac{\partial L}{\partial q_{i0}} = 0$$