Lecture 27  HW 7.8, 7.15, 8.1, 8.9, 9.2, 9.8

Last time

we defined the inertia tensor

$$I_{ij} = \sum_{n} m_n \left( \bar{r}_n^2 \delta_{ij} - \bar{r}_n^i \bar{r}_n^j \right)$$

for a continuous mass distribution this expression is replaced by

$$I_{ij} = \int \rho(\bar{r}) d^3r \left( \bar{r}^2 \delta_{ij} - \bar{r}^i \bar{r}^j \right)$$

the sum of $I_{ij}$ depends on both the origin and the orientation of the coordinate axes. The inertia tensor gives the relation between the angular momentum and the angular velocity

$$J^i = \sum_j I_{ij} \omega^j$$

this same tensor appears in the kinetic energy

$$T = \frac{1}{2} \sum_{n} m_n \frac{\dot{\bar{r}}_n}{\dot{\bar{r}}_n} \cdot \frac{\dot{\bar{r}}_n}{\dot{\bar{r}}_n} =$$

$$= \frac{1}{2} \sum_{n} m_n (\bar{\omega} \times \bar{r}_n) \cdot (\bar{\omega} \times \bar{r}_n)$$
Note
\((\mathbf{\omega} \times \mathbf{r}_n) \cdot (\mathbf{\omega} \times \mathbf{r}_m) =\)
\[\sum_i \varepsilon_{ijk} \varepsilon_{iem} \mathbf{w}_j \mathbf{r}_{ik} \mathbf{w}_e \mathbf{r}_{im} =\]
\[\sum_i (\mathbf{r}_{in}^2 \delta_{ij} \mathbf{w}_i \mathbf{w}_j - \mathbf{w}_i \mathbf{r}_{in} \mathbf{r}_{jm} \mathbf{w}_j) =\]

This gives
\[\mathbf{T} = \frac{1}{2} \sum_i \mathbf{w}_i \mathbf{I}_{ij} \mathbf{w}_j\]

\(\mathbf{I}_{ij}\) is the same tensor that appears in the relation between the angular velocity and angular momentum.

Properties of \(\mathbf{I}_{ij}\)
\[
\mathbf{I}_{ij} = \mathbf{I}_{ji} = \mathbf{I}_{ij}^* \]

It is always possible to find a coordinate system that makes \(\mathbf{I}_{ij}\) diagonal.
In this coordinate system the axes are called the principal axes of inertia.

The diagonal elements are called the principal moments of inertia.

To show this consider the eigenvalue problem

$$\sum (I_{ij} - \lambda s_{ij}) u_j = 0$$

Let $M_{ij}(\lambda) = I_{ij} - \lambda s_{ij}$. If $M_{ij}(\lambda)$ has an inverse then $u_j$ must be 0.

$$0 = \bar{M}^T(\lambda) \bar{M}(\lambda) \bar{u} = \bar{u} = 0$$

The condition for $M(\lambda)$ to not have an inverse is

$$\det M(\lambda) = 0$$

This is a degree 3 polynomial. By the fundamental theorem of algebra this polynomial has 3 roots $\lambda_1, \lambda_2, \lambda_3$. 
For each root there is a solution

\[ \sum I_{ij} U_j(k) = \lambda_k U_i(k) \]

we can use properties of $I_{ij}$ to show

1. The $\lambda_k$ are real
2. The $\lambda_k$ are positive
3. The vectors $\vec{u}(k)$ can be chosen to be real
4. The vectors $\vec{u}(k)$ can be chosen to be orthogonal.

First consider the case that all three roots are distinct

\[ \sum I_{ij} U_j(k) = \lambda_k U_i(k) \]
\[ \sum I_{ij} U_j(l) = \lambda_l U_i(l) = \sum U_j(l) I_{ji} = \lambda_l U_i(l) \]

It follows that

\[ \sum U_i(k) I_{ij} U_j(l) = \lambda_k \sum U_i(k) U_j(l) \]
\[ \sum U_j(l) I_{ij} U_i(k) = \lambda_l \sum U_j(l) U_i(k) \]
This means

\[ \sum (\lambda_k - \lambda_k^*) \sum U_i(k) U_i^*(k) = 0 \]

since \( \lambda_k \neq \lambda_k^* \) we must have \( \sum U_i(k) U_i^*(k) = 0 \)

since \( I_{ii} = I_{ii}^* \)

\[ \sum I_{ii} U_j^*(k) = \lambda_k V_i^*(k) \]

\[ \sum I_{ii} U_j^*(k) = \lambda_k V_i^*(k) = \sum U_j^*(k) I_{ii} \]

It follow that

\[ \sum U_i(k) I_{ii} U_j^*(k) = \lambda_k \sum U_i(k) U_i^*(k) \]

\[ \sum U_i(k) I_{ij} U_i^*(k) = \lambda_k \sum U_i(k) U_i^*(k) \]

subtracting

\[ (\lambda_k - \lambda_k^*) \sum |U_i(k)|^2 = 0 \]

This means \( \lambda_k = \lambda_k^* \)

Finally

\[ I_{ii} U_i(k) = \lambda_k U_i(k) \]

\[ I_{ii} U_i(k)^* = \lambda_k U_i^*(k) \]

This means

\[ I_{ii} (U_i(k) + U_i^*(k)) = \lambda_k (U_i(k) + U_i^*(k)) \]

or

\[ I_{ii} (U_i(k) - U_i^*(k)) = \lambda_k (U_i(k) - U_i^*(k)) \]
both of these are real; at least one is not 0.

For positivity - in this basis
\[ T = \frac{1}{2} \sum \omega_i I_{ii} \omega_i > 0 \]

So in any non 0 \( \mathbf{\tilde{\omega}} \Rightarrow I_{ii} \#(i) > 0 \)

* If two of the eigenvalues are the same then

\[ I (a \mathbf{\tilde{\omega}}(i) + b \mathbf{\tilde{\omega}}(j)) = \lambda (a \mathbf{\tilde{\omega}}(i) + b \mathbf{\tilde{\omega}}(j)) \]

In any \( a \) and \( b \). The constants \( a \) and \( b \) can be chosen to construct pairs of independent orthogonal vectors.

Similarly - if all three eigenvalues are the same then any 3 orthogonal vectors are eigenvectors

given the three orthogonal unit vectors call

\[ \hat{i} = \mathbf{\tilde{\omega}}(1) \]
\[ \hat{j} = \mathbf{\tilde{\omega}}(2) \]
\[ \hat{k} = \mathbf{\tilde{\omega}}(1) \times \mathbf{\tilde{\omega}}(2) = \pm \mathbf{\tilde{\omega}}(3) \]
these are the principal axes of inertia:

\[
\begin{align*}
\mathbf{I} \omega &= \lambda_1 \omega \mathbf{i} \\
\mathbf{I} \omega &= \lambda_2 \omega \mathbf{j} \\
\mathbf{I} \omega &= \lambda_3 \omega \mathbf{k}
\end{align*}
\]

for rotations about the principal axes

\[
\bar{J} = \mathbf{I} \bar{\omega} = I_{zz} \bar{\omega}_k
\]

e tc

\[
T = \frac{1}{2} \omega^2 I_{zz} \quad \text{etc.}
\]

Normally the principal axes can be determined by inspection.

\[
\begin{align*}
I_{ij} &= \int dx \int dy \int dz \left[ \left( x^2 + y^2 + z^2 \right) s_{ij} - r_i r_j \right] \\
&= \text{clearly the off diagonal terms vanish because the integral is odd.}
\end{align*}
\]
\[
\int_{-a/2}^{a/2} x \, dx = \frac{1}{2} \left( \frac{a^3}{4} - \frac{a^3}{4} \right) = 0
\]

etc.

\[
I_{xx} = \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \int_{-c/2}^{c/2} (x^2 + y^2) \, dx \, dy \, dz
\]

\[
= bc \left( \frac{2}{3} \left( \frac{a}{2} \right)^3 \right) + ac \left( \frac{2}{3} \left( \frac{b}{2} \right)^3 \right)
\]

etc.

**Changing the origin**

The inertia tensor depends on both the orientation of the coordinate axes and the origin.

Assume that we have computed the inertia tensor with the origin being the center of mass.

Consider a new origin where the coordinate of the center of mass is \( \bar{R} \).
\[ \bar{\Gamma}_n = \bar{R} + \bar{\Gamma}_{n\text{cm}} \]

where
\[ \sum_{mn} \bar{\Gamma}_n = \sum_{mn} \bar{R} + \sum_{mn} \bar{\Gamma}_{n\text{cm}} \]
\[ \frac{M\bar{R}}{M\bar{R}} \quad 0 \]

\[ I_{ij} = \sum_{mn} \left( (\bar{R} + \bar{\Gamma}_{n\text{cm}})^2 \delta_{ij} - (\bar{R} + \bar{\Gamma}_{n\text{cm}}) (\bar{R} + \bar{\Gamma}_{n\text{cm}}^j) \right) \]

\[ = M(R^2 \delta_{ij} - R_i R_j) + \sum_{mn} \left( \frac{R_{n\text{cm}}^2 \delta_{ij}}{R_{n\text{cm}}^k} - R_{n\text{cm}}^i R_{n\text{cm}}^j \right) \]

\[ + \sum_{mn} \bar{\Gamma}_{n\text{cm}}^i \bar{R} \delta_{ij} \]
\[ - R_i \sum_{mn} \bar{\Gamma}_{n\text{cm}}^i - R_j \sum_{mn} \bar{\Gamma}_{n\text{cm}}^i \]

\[ I_{ij} = I_{ij}^{cm} + M(R^2 \delta_{ij} - R_i R_j) \]

Consider a rigid body that is rotating with angular velocity \( \omega \) about one of the principal axes of inertia (choose coordinates so this axis is \( \hat{R} \))

\[ \bar{J} = I_{22} \bar{\omega} \hat{R} \]
In the absence of external torques, $\dot{\mathbf{J}}$ is conserved.

$$\dot{\mathbf{J}} = 0$$

Next suppose that the system experiences a small torque. Then

$$\dot{\mathbf{J}} = \mathbf{F} \times \mathbf{F}$$

Assume that the force is applied at the axis of rotation.

In this case, $\mathbf{F} \times \mathbf{F}$ is $\bot$ to $\mathbf{\omega}$ and $\mathbf{F}$.

Since $\dot{\mathbf{J}} \bot \mathbf{F}$, in this case the torque will cause the angular momentum to rotate. If we ignore components of the $\mathbf{x}$ momentum to slowly rotate in a direction $\bot$ to $\mathbf{\omega}$ and $\mathbf{F}$. 
Application \( \vec{\omega} = \) 

\[ F = -mg \hat{r} \quad \vec{R} = R \hat{e}_3 \quad (\text{principal axis}) \]

\[ I_3 \omega \hat{e}_3 = -mgR \hat{e}_3 \times \hat{r} \]

The torque is \( \perp \) to \( \vec{\omega} \) so it only changes the direction.

\[ \hat{e}_3 = -\frac{mgR}{I_3 \omega} \hat{e}_3 \times \hat{r} \]

\[ e_{3x} = -\frac{mgR}{I_3 \omega} e_{3y} \]

\[ e_{3y} = -\frac{mgR}{I_3 \omega} (-e_{3x}) \]

\[ e_{3x} = -\left(\frac{mgR}{I_3 \omega}\right) e_{3x} \]

So the principal axis precesses around the direction of the torque with

\[ \omega = \frac{mgR}{I_3 \omega} \]
Consider a rigid body that is free to rotate about a fixed point.

Let \( \hat{e}_1, \hat{e}_2, \hat{e}_3 \) be the principle axes of inertia.

For any point in the body,

\[
\vec{r} = r_1 \hat{e}_1 + r_2 \hat{e}_2 + r_3 \hat{e}_3
\]

where \( r_1, r_2, r_3 \) are constant.

\[
\ddot{\vec{r}} = r_1 \ddot{\hat{e}}_1 + r_2 \ddot{\hat{e}}_2 + r_3 \ddot{\hat{e}}_3
\]

It is always possible to find an instantaneous \( \vec{\omega} \)

define \( a_{ii} = \hat{e}_i \cdot \ddot{\hat{e}}_i \)

\[
\frac{d}{dt} (\hat{e}_i \cdot \ddot{\hat{e}}_i) = \frac{d}{dt} (1) = 0 = 2 \hat{e}_i \dddot{\hat{e}}_i = 2a_{ii} = 0
\]

While \( \hat{e}_i, \hat{e}_j \)

\[
\dddot{\hat{e}}_i = a_{2i} \hat{e}_2 + a_{3i} \hat{e}_3
\]

\[
0 = \frac{d}{dt} (\hat{e}_i \cdot \dddot{\hat{e}}_i) = a_{2i} + a_{3i}\
\]
This means that $a_{ij}$ is an antisymmetric matrix:

$$a_{ij} = \begin{pmatrix} 0 & a_{i2} & a_{i3} \\ -a_{i2} & 0 & a_{i3} \\ -a_{i3} & -a_{i2} & 0 \end{pmatrix}$$

Define $\omega$ as:

$$\omega_1 = -a_{13}$$
$$\omega_2 = -a_{31}$$
$$\omega_3 = -a_{12}$$

Then:

$$\dot{e}_1 = a_{21} e_2 + a_{31} e_3 = \omega_3 e_2 - \omega_2 e_3 = (\bar{\omega} \times \dot{e}),$$

In general:

$$\dot{e}_i = \bar{\omega} \times \dot{e}_i.$$

This shows that the instantaneous angular velocity always exists.
general equations

\[ \ddot{\bar{r}}_n = \bar{R} + \ddot{\bar{r}}_{n\text{cm}} \]

\[ \dot{\bar{r}}_n = \dot{\bar{R}} + \dot{\bar{r}}_{n\text{cm}} = \dot{\bar{R}} + \bar{\omega} \times \ddot{\bar{r}}_{n\text{cm}} \]

\[ \dot{\bar{p}} = M \ddot{\bar{R}} = \sum F_i \]

\[ \dot{\bar{J}}_{n\text{cm}} = \sum \bar{r}_{n\text{cm}} \times \ddot{\bar{r}}_{n\text{cm}} \]

\[ \bar{J}_{n\text{cm}} = I_{xx} \bar{w}_1 \hat{e}_1 + I_{yy} \bar{w}_2 \hat{e}_2 + I_{zz} \bar{w}_3 \hat{e}_3 \]

because the principal axes rotate in an inertial coordinate system, it is sometimes useful to consider the equations of motion in the body-fixed system.

\[ \frac{d\bar{J}_b}{dt} = \frac{d\bar{J}_b}{dt} + \bar{\omega} \times \bar{J}_b = \bar{\sigma} = \text{tang} \]

in the body-fixed system

\[ \frac{d\bar{J}_b}{dt} = I_1 \bar{w}_1 \hat{e}_1 + I_2 \bar{w}_2 \hat{e}_2 + I_3 \bar{w}_3 \hat{e}_3 \]

since \( I_i \hat{e}_i \) are fixed in the body
These equations become

\[ I_x \dot{\omega}_x + \omega_y I_z \omega_z - \omega_z I_y \omega_y = \mathbf{g}_x \]
\[ I_y \dot{\omega}_y + \omega_z I_x \omega_x - \omega_x I_z \omega_z = \mathbf{g}_y \]
\[ I_z \dot{\omega}_z + \omega_x I_y \omega_y - \omega_y I_x \omega_x = \mathbf{g}_z \]

These equations are called Euler's equations. The advantage is that the principle moments are fixed; but \( \mathbf{g}_x \) must be transformed to the moving system.