General Lagrangian dynamics

0. \( (m_n \frac{d^2 \tilde{r}_n}{dt^2} - \tilde{F}_n^A - \tilde{F}_n^c) = 0 \)

3. \( h_k(q_1, q_{3n}, t) = 0 \) holonomic constraint

0 = 2 \( \frac{\partial h_k}{\partial q_e} \) \( s q_e \)

linear relation among constrained generalized coordinates

use these linear relations to eliminate dependent \( s q_k \)

3. \( 0 = \sum_{n=1}^{3n} (m_n \frac{d^2 \tilde{r}_n}{dt^2} - \tilde{F}_n^A - \tilde{F}_n^c) \cdot \delta \tilde{r}_n \)

choose \( \delta \tilde{r}_n \) consistent with constraint \( (\delta \tilde{r}_n \cdot \tilde{F}_n^c = 0) \)

\( \delta \tilde{r}_n = \sum_{l=1}^{3n-k} (\frac{\partial \tilde{r}_n^l}{\partial q_e}) \) \( s q_e \)

0 = 2 \( \frac{\partial c}{\partial q_e} \) \( \sum_{n=1}^{3n} (m_n \frac{d^2 \tilde{r}_n}{dt^2} - \tilde{F}_n^A) \cdot \frac{\partial \tilde{r}_n}{\partial q_e} \)

use \( \frac{\partial r}{\partial q_e} = \frac{\partial \tilde{r}_n}{\partial q_e} \) and \( \frac{d}{dt} (\frac{\partial r}{\partial q_k}) = \frac{\partial}{\partial q_k} \frac{\partial r}{\partial q_k} \)

0 = \sum_{k=1}^{3n} \left[ \left( \frac{d}{dt} \frac{\partial r}{\partial q_k} - \frac{\partial}{\partial q_k} \right) T - \tilde{F}_n^A \frac{\partial \tilde{r}_n}{\partial q_k} \right] \) \( s q_k \)
Since the $\delta q_k$ are independent, we get

$$\frac{d}{dt} \left( \frac{\partial T}{\partial q_k} \right) - \frac{\partial T}{\partial q_k} = F_k$$

$$F_k = \sum \overline{F}_n \cdot \frac{\partial \overline{r}_n}{\partial q_k} \quad \text{k'th generalized force}$$

* This method works when the forces are not conservative.

* When

$$F_k = \frac{d}{dt} \left( \frac{\partial \mathbf{U}}{\partial \dot{q}_k} \right) - \frac{\partial \mathbf{U}}{\partial q_k}$$

true for conservative forces, also true for some other forces.

* This method justifies ignoring constraints and just working with independent generalized coordinates.
examples: symmetric top - this time with a gravitational term

\[
L = \frac{1}{2} I_1 \sin^2 \theta \dot{\phi}^2 + \frac{1}{2} I_3 \dot{\theta}^2 + \frac{1}{2} I_3 (\dot{\phi} + \dot{\phi} \cos \theta)^2 - mg R \cos \theta
\]

New term

here \( \theta, \phi, \dot{\phi} \) are the generalized coordinates of the top

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = 0 \quad \Rightarrow \quad I_1 \sin^2 \theta \dot{\phi} + I_3 (\dot{\phi} + \dot{\phi} \cos \theta) \cos \theta = c_1
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \theta} \right) = 0 \quad \Rightarrow \quad I_3 (\dot{\phi} + \dot{\phi} \cos \theta) = c_2
\]

where \( c_1 \) and \( c_2 \) are constants. Using the second equation in the first we give:

\[
I_1 \sin^2 \theta \dot{\phi} + c_2 \cos \theta = c_1
\]

\[
\dot{\phi} = \frac{c_1 - c_2 \cos \theta}{I_1 \sin^2 \theta}
\]

using energy conservation gives:

\[
E = \frac{1}{2} I_1 \sin^2 \theta \left( \frac{c_1 - c_2 \cos \theta}{I_1 \sin^2 \theta} \right)^2 + \frac{1}{2} I_3 \dot{\theta}^2 + \frac{1}{2} I_3 \dot{\phi}^2 + \frac{1}{2} \frac{c_2^2}{I_3} + mg R \cos \theta
\]

\[
\dot{\theta}^2 = - \left( \frac{c_1 - c_2 \cos \theta}{I_1 \sin^2 \theta} \right) - \frac{c_2^2}{I_3 \dot{\phi}^2} - \frac{2 mg R \cos \theta}{I_1} + \frac{2 E}{I_1}
\]

\[
\dot{\phi}^2 = \frac{2E}{I_1} - \left( \frac{c_1 - c_2 \cos \theta}{I_1 \sin^2 \theta} \right) - \frac{c_2^2}{I_3 \dot{\phi}^2} - \frac{2 mg R \cos \theta}{I_1}
\]
* The right side of the \( \Theta \) equation must be non-negative —
  
  \[ \Theta = \frac{2E}{I_1} (1 - \cos^3 \Theta) + \left( \frac{c_1 - c_2 \cos \Theta}{I_1} \right)^2 - \frac{c_3^2}{I_1 I_3} (1 - \cos^2 \Theta) \]
  
  - \( \frac{2mqR}{I_1} \cos \Theta (1 - \cos^2 \Theta) \)

  \( \Theta \) is a 3rd degree polynomial in \( \cos \Theta \)

  \[ \Theta = \frac{2mqR}{I_1} \cos^3 \Theta + \cos \Theta \left( \frac{c_1^2}{I_1^2} + \frac{c_2^2}{I_3^2} - \frac{2E}{I_1} \right) \]
  
  \[ - \left( \frac{2c_1 c_2}{I_1^2} - \frac{2mqR}{I_1} \right) \cos \Theta + \left( \frac{c_1^2}{I_1^2} - \frac{c_3^2}{I_3^2} + \frac{2E}{I_1} \right) \]

  The only roots that are physical are real roots between \(-1, 1\).

  \[
  \\therefore \text{If there is 1 root then } \Theta = \text{constant is a solution. If there are 2 real roots then we expect that } \hat{e}_3 \text{ oscillates between these roots.}
  \]
when θ varies it is possible that ϕ changes sign.

The inflection points can be determined by differentiating the polynomial and setting the results to 0.

Example 2 - pendulum on massless rod that rotates with constant angular velocity.

\[
\dot{\omega} = \frac{1}{2} m (\ell \ddot{\theta} + \ell \dot{\theta}^2 \sin \phi \cos \omega) + mg \ell \cos \omega
\]

where \( \phi = \omega \) is fixed. Then there is one degree of freedom.

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0
\]

\[
\frac{\partial L}{\partial \theta} = \frac{1}{2} m \ell \omega^2 - m g \ell \sin \theta
\]

\[
\frac{\partial L}{\partial \dot{\theta}} = m \ell \dot{\theta}
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) = m \ell^2 \ddot{\theta}
\]

\[
m \ell^2 \ddot{\theta} - \frac{1}{2} m \ell \omega^2 \sin \theta \cos \phi + m g \ell \sin \theta
\]
while energy is not conserved, we still get a conservation law

multiply by $\dot{\phi}$

$$m\ell^2 \dot{\phi} \ddot{\phi} - \frac{3}{8} m \omega^2 \sin \phi \cos \phi \dot{\phi} + m q \ell \sin \phi \dot{\phi}$$

$$\frac{d}{dt} \left( \frac{1}{2} m \ell^2 \dot{\phi}^2 - \frac{1}{2} m \omega^2 \sin^2 \phi - m q \ell \cos \phi \right)$$

this gives

$$C = \frac{1}{2} m \ell^2 \dot{\phi}^2 - \frac{1}{2} \omega^2 m \sin^2 \phi - m q \ell \cos \phi$$

we can consider

$$U_{eff} = -\frac{1}{2} \omega^2 m \sin^2 \phi - m q \ell \cos \phi$$

we can look for equilibrium points

when $\frac{\partial U_{eff}}{\partial \phi} = 0$

$$-\omega^2 m \sin \phi \cos \phi + m q \ell \sin \phi = 0$$

$$\sin \phi (\ell m q - \omega^2 m \cos \phi)$$

this has 2 roots

$\phi = 0, \pi$

$$\cos \phi = \frac{q}{\ell \omega^2} \quad \text{(valid for } \frac{q}{\ell} < \omega^2)$$
we can check the stability

\[ \cos \omega (mqe - \ell^2 \omega^2 m \cos \theta) + \ell^3 \omega \sin^2 \theta = \]

\[ -mqe \cos \omega + \ell^3 \omega \sin (\sin^2 \theta - \cos^2 \theta) \]

\[ \text{In } \theta = 0 \quad \frac{\partial \nu}{\partial \theta} = m q e - m q e \omega^2 \quad (\text{stable if } \frac{q}{e} - \omega^2 < 0) \]

\[ \text{(unstable if } \frac{q}{e} - \omega^2 < 0) \]

\[ \text{In } \theta = \pi \quad \frac{\partial \nu}{\partial \theta} = -mq e - m e \omega^2 < 0 \quad \text{unstable} \]

\[ \text{In } \cos \theta = \frac{q}{e \omega} \]

\[ \frac{\partial \nu}{\partial \omega} = \frac{q}{e \omega} m q e + \ell^3 \omega \left( 1 - 2 \frac{q^2}{e^2 \omega^2} \right) \]

\[ = \ell^3 \omega m - m \frac{q^2}{\omega^2} \]

\[ = \ell^3 \omega (1 - \left( \frac{q}{e \omega} \right)^2) > 0 \quad \text{if } \frac{q}{e \omega} < 1 \]

stable

Method of Lagrange multipliers

\[ H_i(q_1, q_2, t) = 0 \quad \text{constraint } \Rightarrow \]

\[ \sum \frac{\partial H_i}{\partial q_k} q_k = 0 \]

rather than eliminate dependent variables consider
\[ L + 2 \lambda \kappa h \kappa \]

undetermined quantities

\[
\sum \int_{t_i}^{t_f} \left( \sum \frac{\partial L}{\partial q_i} \dot{q}_i + \sum \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i + \sum \lambda_k \frac{\partial h}{\partial \dot{q}_k} \dot{q}_k \right) = 0
\]

\[
\sum \int_{t_i}^{t_f} \left( \sum \frac{\partial L}{\partial q_i} \dot{q}_i - \sum \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \dot{q}_i + \sum \lambda_k \frac{\partial h}{\partial \dot{q}_k} \dot{q}_k \right) = 0
\]

\[
0 \rightarrow \text{vanishes at endpoints}
\]

In this case the \( q_k \) are not independent
(we have \( 3N \) \( q_i \) and \( M \) constraints and \( 3N - M \) independent \( q_i \))

\[ x \text{ choose } \lambda_i, \lambda_m \text{ so the coefficients of } q_i, \dot{q}_m \text{ vanish} \]

\[ \text{the remaining } q \text{ are independent} \]

This gives
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 2 \lambda \kappa \frac{\partial h_i}{\partial q_i} = \dot{f}_i \]

The \( \lambda_i \) are generalized forces due to constraint.

\[ r - R = h = 0 \]

\[ L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - m g r \cos \phi + \lambda h(r) \]

Equations:

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = 0 = m \dot{r} - m r \dot{\phi}^2 + m g \cos \phi - \lambda \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} = \frac{d}{dt} (m r^2 \dot{\phi}) - m g r \sin \phi = 0 \]

Imposing the constraint:

\[ \lambda = -m r \dot{\phi}^2 + m g \cos \phi \]

\[ m r^2 \ddot{\phi} - m g r \sin \phi = 0 \]

Multiply by \( \dot{\phi} \):

\[ m r^2 \dot{\phi} \dot{\phi} - m g r \sin \phi \dot{\phi} = 0 \]

\[ \frac{d}{dt} \left( \frac{1}{2} m r^2 \dot{\phi}^2 + m g r \cos \phi \right) = 0 \]

\[ E = \frac{1}{2} m r^2 \dot{\phi}^2 + m g R \cos \phi \]

At the top: \( E = m g R \)
\[ \frac{1}{2} m R^2 \dot{\theta}^2 + m q R \cos \theta = m q R \]
\[ m R^2 \dot{\theta}^2 = 2 m q R (1 - \cos \theta) \]
\[ -m R \dot{\theta}^2 = -2 m g (1 - \cos \theta) \]

Using this in \( \lambda \)

\[ \lambda = -2 m g (1 - \cos \theta) + m q \cos \theta \]
\[ = -2 m g + 3 m q \cos \theta \]

Setting \( \lambda = 0 \implies \]

\[
\begin{bmatrix}
\cos \theta = \frac{2}{3}
\end{bmatrix}
\]

This is the angle where the radial component of the force vanishes.
special case - charged particle in an electromagnetic field

$$\vec{F} = q \left( \vec{E} + \vec{v} \times \vec{B} \right)$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{E} = -\vec{v} \phi - \frac{\partial \vec{A}}{\partial t}$$

Let

$$\vec{v} = q \vec{\phi} - q \vec{r} \cdot \vec{A}$$

$$-\frac{\partial \vec{v}}{\partial x} = -q \frac{\partial \phi}{\partial x} + q \left( x \frac{\partial A_x}{\partial x} + y \frac{\partial A_y}{\partial x} + z \frac{\partial A_z}{\partial x} \right)$$

$$\frac{d}{dt} \left( \frac{\partial \vec{v}}{\partial x} \right) = -q \left( \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x} x + \frac{\partial A_y}{\partial x} y + \frac{\partial A_z}{\partial x} z \right)$$

Adding these

$$\frac{d}{dt} \left( \frac{\partial \vec{v}}{\partial x} \right) - \frac{\partial \vec{v}}{\partial x} =$$

$$-q \left( \frac{\partial A_y}{\partial t} + \frac{\partial \phi}{\partial x} \right) + q \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) + \frac{z}{c^2} \left( \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right)$$

$$= q \vec{E_x} + q \left( \nabla \times \vec{B} \right)$$

Similar results are obtained in the other 3 components:

$$L = \frac{1}{2} m \vec{v}^2 - q \phi + q \vec{r} \cdot \vec{A}$$

Lagrangian equations give the correct equations of motion.
Transverse oscillations of a string

Let $y(x,t)$ be the transverse displacement of the string at time $t$ and position $x$.

for a part of length $dx$ at $x$, the kinetic energy is

$$dT = \frac{1}{2} \rho \frac{m}{L} dx \left( \frac{dy}{dt} \right)^2$$

the potential energy of the string,

$$dV = dw = F dx \quad (F = \text{Tension})$$

$$= \sqrt{dx^2 + (dy)^2} \quad F = F \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dx$$

$$= F \left( 1 + \frac{1}{2} \left( \frac{dy}{dx} \right)^2 \right) dx$$

In small displacement

$$L = \int_0^L dx \left( \frac{1}{2} \rho \left( \frac{dy}{dt} \right)^2 - \frac{F}{2} \left( \frac{dy}{dx} \right)^2 \right) dx$$

$$A = \int_{t_1}^{t_2} dt \int_0^L dx \left( \frac{1}{2} \rho \left( \frac{dy}{dt} \right)^2 - \frac{F}{2} \left( \frac{dy}{dx} \right)^2 - F \right) dx$$

If $y_0(x,t)$ is the solution with

$y(x,t_1) = y_1(x)$

$y(x,t_2) = y_2(x)$

$y(0,t) = g_0(t)$

$y(L,t) = g_L(t)$
of all the possible \( y(x,t) \) that satisfy these 4 boundary + initial and final conditions we look for one that makes the action stationary

\[
y(x,t) = y_0(x,t) + \eta \delta y(x,t)
\]

\[
\delta y(0,t) = \delta y(L,t) = \delta y(x,t_0) = 0
\]

\[
SA = \frac{\partial}{\partial m} \int dx \int dt \left( \frac{1}{2} \rho \left( \frac{\partial y_0}{\partial t} \frac{\partial \delta y}{\partial t} \right)^2 - \frac{1}{2} F \left( \frac{\partial y_0}{\partial x} + \eta \frac{\partial \delta y}{\partial x} \right)^2 \right)
\]

\[
\int dx \int dt \left[ \rho \frac{\partial y_0}{\partial t} \frac{\partial \delta y}{\partial t} - F \frac{\partial y_0}{\partial x} \frac{\partial \delta y}{\partial x} \right] = 0
\]

\[
\int dx \int dt \left[ - \rho \frac{\partial}{\partial t} \left( \frac{\partial y_0}{\partial t} \right) + F \frac{\partial}{\partial x} \left( \frac{\partial y_0}{\partial x} \right) \right] \delta y + \sum_{i=0}^{L} \int dx \int dt \frac{\partial}{\partial t} \left( \frac{\partial y_0}{\partial t} \delta y \right) + \sum_{i=0}^{t} \int dx \int dt \frac{\partial}{\partial x} \left( \frac{\partial y_0}{\partial x} \delta y \right)
\]

The 2 terms vanish because \( \delta y(x_0) = \delta y(x_L) = 0 \) and \( \delta y(L,t) = \delta y(t_0) = 0 \). Since \( \delta y \) is arbitrary we get

\[
\left( \frac{\partial^2 y_0}{\partial t^2} - \frac{F}{\rho} \frac{\partial^2 y_0}{\partial x^2} \right) = 0
\]

This is called the wave equation

\[
C^2 = \frac{F}{\rho}
\]

has units of \((\text{velocity})^2\). It is the
\( c \) is the speed of a wave in the string.

Let \( y = f(x \pm ct) \)

\[
\frac{\partial^2 y}{\partial x^2} = f''
\]

\[
\frac{\partial^2 y}{\partial t^2} = c^2 f''
\]

which gives

\[
c^2 \frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = 0
\]

so any function of \( x \pm ct \) is a solution of this equation.

This example shows that Lagrangian methods can be applied to systems with an infinite number of degrees of freedom.