Fields

- $T(\vec{r},t)$ temperature (scalar field)
- $\vec{v}(\vec{r},t)$ wind velocity (vector field)
- $F^{\mu\nu}(\vec{r},t)$ electromagnetic field (tensor field)

String (2 dimensions)

\[ y(x^1) \]

\[ F = \text{tension} \]

Kinetic energy

\[ T = \sum_i \frac{1}{2} \left( \frac{m}{L} \right) \left( \frac{\partial y}{\partial t} (x_i, t) \right)^2 \to \frac{1}{2} \int_0^L \rho \left( \frac{\partial y}{\partial t} \right)^2 dx \]

Potential

\[ dV = \vec{F} \cdot d\vec{s} = F \sqrt{1 + \left( \frac{\partial y}{\partial x} \right)^2} dx \]

\[ V = \int_0^L F \left( 1 + \frac{1}{2} \left( \frac{\partial y}{\partial x} \right)^2 \right) dx \]

\[ V \approx F L + \int_0^L \frac{F}{2} \left( \frac{\partial y}{\partial x} \right)^2 dx \]

\[ L = T - V = \int_0^L \left( \frac{1}{2} \left( \rho \left( \frac{\partial y}{\partial t} \right)^2 - F \left( \frac{\partial y}{\partial x} \right)^2 \right) \right) dx \]

We define the action

\[ A = \int_0^T L(\vec{r},t') dt' \int_0^T dt \int_0^L dx \left( \frac{1}{2} \left( \rho \left( \frac{\partial y}{\partial t} \right)^2 - F \left( \frac{\partial y}{\partial x} \right)^2 \right) + \text{constant} \right) \]
A is a functional of a function $y(x,t)$. Let $y_0(x,t)$ be the function that makes the action extremal

$$y(x,t) = y_0(x,t) + \lambda sy(x,t)$$

We look for function that make this extremal subject to

$$y(0,t) = f_0(t) \quad y(x,0) = q_0(x)$$
$$y(L,t) = f_L(t) \quad y(x,T) = q_T(x)$$

This requires

$$sy(0,t) = sy(L,t) = 0$$
$$sy(x,0) = sy(x,T) = 0$$

$$0 = SA(t) = \frac{d}{dx} A \{ y_0 + \lambda sy \}$$

for all $sy$

$$\lambda = 0$$

$$\int_0^T \int_0^L \left( \rho \frac{\partial y_0}{\partial t} \frac{\partial sy}{\partial x} - F \frac{\partial y_0}{\partial x} \frac{\partial sy}{\partial x} \right) dx dt =$$

$$\int_0^T \int_0^L \left( - \frac{\partial}{\partial t} \left( F \frac{\partial y_0}{\partial t} \right) sy + \frac{\partial}{\partial x} \left( F \frac{\partial y_0}{\partial x} sy \right) \right) dx dt +$$

$$\int_0^T \int_0^L \left( \frac{\partial}{\partial x} \left( \rho \frac{\partial y_0}{\partial t} sy \right) - \frac{\partial}{\partial y} \left( F \frac{\partial y_0}{\partial x} sy \right) \right) dx dt =$$

$$\int_0^T \int_0^L \left( - \rho \frac{\partial^2 y_0}{\partial t^2} + F \frac{\partial y_0}{\partial x} \right) sy \right) dx dt +$$

$$\int_0^T \int_0^L \left( \rho \frac{\partial y_0}{\partial t} sy \right) \bigg|^T_0 + \int_0^T \frac{\partial y_0}{\partial x} sy \bigg|^L_0$$
For this to vanish for all \( y \):

\[-\rho \frac{\partial^2 y}{\partial t^2} + F \frac{\partial^2 y}{\partial x^2} = 0\]

\[\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = 0 \quad c = \frac{F}{\rho}\]

This is called the wave equation;

\( c \) is a constant with dimensions of speed.

Solutions:

Let \( f(x) \) be a function with 2 derivatives:

\[y = f(x \pm ct)\]

\[\frac{\partial^2 y}{\partial t^2} = c^2 f'' \quad \frac{\partial^2 y}{\partial x^2} = f''\]

\[\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = (c^2 - c^2) f'' = 0\]

These are solutions of the wave equation for any \( f(x) \).

Let \( y(x,t) = f(x+ct) + g(x-ct) \) -

this is obviously a solution.

Consider a string that is clamped down at both ends so \( y(0,t) = y(L,t) = 0 \).
This gives

\[ y(0t) = 0 = f(ct) + g(-ct) \]
\[ g(-ct) = -f(ct) \]
\[ g(x) = -f(-x) \]

Then

\[ y(xt) = f(x-ct) - f(-(x+ct)) \]
\[ = f(x-ct) - f(-x-ct) \]

For the other boundary condition

\[ y(Lt) = f(L-ct) - f(-L-ct) = 0 \]

Let

\[ z = L-ct \quad -ct = z-L \]
\[ f(z) - f(-L+z-L) = f(z) - f(z-2L) = 0 \]

This shows that \( f(z) \) is periodic with period 2L.

A periodic function with period 2L

\[ f(z) = \sum_{n=1}^{\infty} a_n \sin \left( \frac{2\pi n z}{L} \right) + \sum_{n=1}^{\infty} b_n \cos \left( \frac{2\pi n z}{L} \right) \]

or

\[ \sum_{n=1}^{\infty} c_n e^{\frac{2\pi i n z}{L}} \quad c_n \text{ complex} \]

Both expansions are equivalent.
The text on the image is as follows:

\[ y(x_0) = q_0(x) = \]
\[ \sum a_n \sin \left( \frac{2\pi n}{2L} x \right) = \sin \frac{2\pi n}{2L} (-x) \]
\[ \sum b_n \cos \left( \frac{2\pi n}{2L} x \right) = \cos \frac{2\pi n}{2L} (-x) \]
\[ 2 \sum a_n \sin \left( \frac{2\pi n}{2L} x \right) \]

Multiply by \( \sin \left( \frac{2\pi m y}{2L} \right) \) and integrate from 0 to 2L:
\[ \int \sin \left( \frac{2\pi m y}{2L} \right) g_0(x) = \sum \frac{s_{mn}}{m} 2a_n \frac{1}{2L} 2L = 2a m L \]
\[ a_m = \frac{1}{2L} \int_0^{2L} \sin \left( \frac{\pi m y}{L} \right) g_0(x) \, dx \]

Note:
\[ \sin \left( \frac{\pi m y}{L} \right) \sin \left( \frac{\pi n y}{L} \right) = \]
\[ \frac{1}{2} \left( \cos \left( \frac{\pi (m-n) y}{L} \right) - \cos \left( \frac{\pi (m+n) y}{L} \right) \right) \]

These integrals to \( a_m \) for \( m \neq n \) are given that the \( a_n \) are known, the final condition:
\[ y(x_T) = q_T(x) \]
\[ \sum a_n \sin \left( \frac{\pi n}{L} (x-T) \right) = \sin \left( \frac{\pi n}{L} (-x-T) \right) \]
\[ + \sum b_n \cos \left( \frac{\pi n}{L} (x-T) \right) = \cos \left( \frac{\pi n}{L} (-x-T) \right) \]
Note \( \frac{\pi x}{L} = \phi \) is a phase.

In this case,

\[
\sin \left( \frac{\pi y}{L} - \phi \right) - \sin \left( \frac{\pi y}{L} + \phi \right) = \\
\sin \left( \frac{\pi x}{L} - \phi \right) + \sin \left( \frac{\pi x}{L} + \phi \right) = \\
2 \sin \left( \frac{\pi x}{L} \right) \cos \phi \\
\cos \left( \frac{\pi y}{L} - \phi \right) - \cos \left( \frac{\pi y}{L} + \phi \right) = \\
2 \sin \left( \frac{\pi y}{L} \right) \sin \phi
\]

This gives

\[
g_r(x) = 2 \sin \left( \frac{\pi x}{L} \right) \left( a_n \cos \phi + b_n \sin \phi \right)
\]

\[
\frac{1}{2L} \int_0^{2L} g_r(x') \sin \left( \frac{\pi y}{L} \right) dx' = 1 \left( a_m \cos \phi + b_m \sin \phi \right)
\]

\[
b_m \sin \phi = \frac{1}{2L} \int_0^{2L} g_r(x') \sin \left( \frac{\pi y}{L} \right) dx' - a_m \cos \phi
\]

This expresses all of the unknown coefficients \( a_n \) and \( b_n \) in terms of the functions \( g(x_0) \) and \( g(x_T) \).
Small oscillations about stable equilibrium

Assume

$$\ddot{\vec{r}}_n = \ddot{\vec{r}}_0 (q_1, q_k)$$

$$V = V(q_1, q_k)$$

$$T = \sum \frac{1}{2} m_n \ddot{r}_n \cdot \ddot{r}_n$$

$$\ddot{r}_n = \sum \frac{\partial r_n}{\partial q_k} \dot{q}_k$$

$$T = \sum \frac{1}{2} m_n \ddot{r}_n \cdot \ddot{r}_n \dot{q}_k \dot{q}_e$$

this has the general term

$$T = \frac{1}{2} \sum \dot{q}_k M_{re}(\ddot{q}) \dot{q}_e$$

$$M_{re} = \sum m_n \frac{\partial \vec{r}_n \cdot \vec{r}_e}{\partial q}$$

Properties of $M$

1. $M_{re} = M_{re}^x \text{ real}$

2. $M_{re} = M_{re}^x$

3. $\sum M_{re} q_k q_e > 0$ (kinetic energy > 0)
double Taylor series

\[ f(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x_0, y_0)(x - x_0)^2 + \ldots \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

\[ f(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) + \frac{1}{2} \frac{\partial^2 f}{\partial y^2}(x_0, y_0)(y - y_0)^2 + \ldots \]

\[ \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)(x - x_0)(y - y_0) + \frac{1}{2} \frac{\partial^3 f}{\partial x^3}(x_0, y_0)(x - x_0)^3 + \ldots \]

retaining terms that are up to second order in \((x-x_0), (y-y_0)\) give

\[ f(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x_0, y_0)(x - x_0)^2 \]

\[ + \frac{1}{2} \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)(x - x_0)(y - y_0) + \frac{1}{2} \frac{\partial^2 f}{\partial y^2}(x_0, y_0)(y - y_0)^2 \]

+ \( O(\Delta^3) \)

This generalizes to any number of variables

\[ f(q_1, q_k) = f(q_{10}, q_{k0}) + \sum_{i} \frac{\partial f}{\partial q_{i0}}(q_{i0}) (q_i - q_{i0}) \]

\[ + \frac{1}{2} \sum_{i} \frac{\partial^2 f}{\partial q_{i0}^2} (q_i - q_{i0}) (q_i - q_{i0}) + \ldots \]
A point $\bar{q}_o = (q_{1o}, q_{Ko})$ is called an equilibrium point if

$$F_k = -\frac{\partial V}{\partial q_k}(\bar{q}_o) = 0$$

This means that all of the generalized forces vanish at an equilibrium point.

For small $(\bar{q} - \bar{q}_o)$ we can write

$$V(\bar{q}) = V(\bar{q}_o) + \frac{1}{2} \sum \frac{\partial^2 V}{\partial q_m \partial q_n}(\bar{q}_o) (q_m - q_{mo})(q_n - q_{no}) + O(\Delta q^2)$$

The first term is a constant that is independent of the physics (it does not contribute to Lagrange equations).

The second term vanishes because of the equilibrium condition:

$$V(\bar{q}) = \frac{1}{2} \sum V_{mn} (q_m - q_{mo})(q_n - q_{no})$$

$$V_{mn} = \frac{\partial^2 V}{\partial q_m \partial q_n}(\bar{q}_o)$$
The equilibrium point $q_0$ is called stable if the potential increases for any small displacement:

$$\sum V_{mn} \delta m \delta n > 0$$

as with the matrix $M_{mn}$,

$$V_{mn} = V_{nm} = V_{mn} > 0$$

Conditions for a matrix to be positive. Consider

$$\det (V_{mn} - \lambda I_{mn}) = 0$$

This is a degree $K$ polynomial in $\lambda$. It has $K$ roots, we

$$\lambda_1, \lambda_2, \ldots, \lambda_K$$

$$V_{mn} u_m^\lambda = \lambda \cdot u_m^\lambda$$

eigenvalues

$$u_m^\lambda V_{mn} u_n^\lambda = \lambda \cdot \sum u_m^\lambda \cdot u_n^\lambda$$

$$u_n^\lambda V_{mn} u_m^\lambda = \lambda \cdot \sum u_n^\lambda \cdot u_m^\lambda$$
If $r = k$, then $\lambda_k = \lambda_k^*$, the eigenvalues must be real. If $r \neq k$ and $\lambda_k \neq \lambda_k^*$, then $\sum_{m} \overline{u}_m^* u_m^k = 0$.

The $u_k^k$ can all be chosen to be real.

\[ \overline{u}_k^k = \lambda_k^* \overline{u}_k^k \]
\[ u_k^k = \lambda_k^* u_k^k \]

\[ \overline{u}_k^k \overline{u}_k^k \] can be replaced by \[ u_k^k + u_k^k \]
both of which are real. Finally, if a set of eigenvalues are identical, they can be chosen to be orthonormal.

\[ u_k^k \] are independent orthonormal vectors.

A general vector has an expansion of the form

\[ \overline{\xi} = \sum_{k} c_k \overline{u}_k^k \]

\[ \sum_{m} \overline{v}_m^k \overline{v}_m^k = \sum_{k} c_k \overline{v}_m^k \overline{v}_m^k \lambda_k \]

Since $c_k^2 > 0$, $\xi$ is a non-zero vector.
This will be positive if and only if all of the eigenvalues of $V_{ij}$ are positive.

Next we introduce new generalized coordinates

$$\mathbf{\eta}_i = \mathbf{q}_i - \mathbf{q}_0, \quad \dot{\mathbf{\eta}}_i = \dot{\mathbf{q}}_i$$

(displacements from equilibrium)

we consider the case where

1. there is an equilibrium point $\mathbf{q}_0$,

2. the equilibrium point is stable

$$V_{ij} = \frac{3}{2}V_{\mathbf{q}_i \mathbf{q}_j} (\mathbf{q}_0)$$

has positive eigenvalues

3. $M_{ij}(\mathbf{q}) = M_{ij}(\mathbf{q}_0) + 2 \frac{\partial^2 M_{ij}}{\partial \mathbf{q}_i \partial \mathbf{q}_j} (\mathbf{q}_0) (\mathbf{q} - \mathbf{q}_0) + \cdots$ $M_{ij}(\mathbf{q}_0)$ is a constant positive matrix $\mathbf{M}_{ij}$

$$T = \frac{1}{2} 2 \sum_i M_{ij} (\mathbf{q}_0) \dot{\mathbf{q}}_i \dot{\mathbf{q}}_j + \frac{1}{2} \sum_i \sum_{j 
eq k} \frac{\partial^2 M_{ij}}{\partial \mathbf{q}_i \partial \mathbf{q}_j} (\mathbf{q}_0) \dot{\mathbf{q}}_i \dot{\mathbf{q}}_j (\mathbf{q} - \mathbf{q}_0) \dot{\mathbf{q}}_k + \cdots$$

$\sim \mathbf{\eta} \dot{\mathbf{\eta}}$
To second order in $\eta_i, \dot{\eta}_i$

$$L = T - V = \sum \frac{1}{2} \eta_i \dot{M}_{ij} \dot{\eta}_j - \frac{1}{2} \sum \eta_i V_{ij} \eta_j + o(\eta^3)$$

For small displacements from stable equilibrium we can ignore the $\eta^3$ terms

(mis is not justified if the equilibrium is not stable)

Equations of motion

$$\frac{\partial L}{\partial \dot{\eta}_i} = \sum M_{ij} \ddot{\eta}_j$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\eta}_i} \right) = \sum M_{ij} \ddot{\eta}_j$$

$$\frac{\partial L}{\partial \eta_i} = -V_{ij} \eta_j$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\eta}_i} \right) - \frac{\partial L}{\partial \eta_i} = \sum M_{ij} \ddot{\eta}_j + \sum V_{ij} \eta_j = 0$$

This is a system of $K$ coupled ODEs with constant coefficients.
since both $M$ and $V$ are positive, the determinant = product of eigenvalues $\neq 0$ means $M$ has an inverse

$$\frac{d^2 \tilde{n}}{dt^2} = M^{-1} V \tilde{n}$$

from what we learned earlier

$$-M^{-1} V \tilde{n} = \tilde{n}(t)$$

is a solution to this equation. This can be checked by inserting this expression into the differential equation.

We will not do that - instead, we will look for solutions of the form

$$\tilde{n}(t) = \tilde{n}(0) e^{-i\omega t}$$

using this expression in the differential equation gives

$$\Sigma (-\omega^2 M_{ij} + V_{ij}) \tilde{n}_j(0) e^{-i\omega t} = 0$$
This is called a generalized eigenvalue problem. It can only have non-zero solution when

$$\det (-\omega^2 M + V) = 0$$

This gives k roots for $\omega^2$

eigenvalues satisfy

$$\omega_n^2 M \mathbf{u}_n^{(m)} = V \mathbf{u}_n^{(m)}$$

$$\omega_n^2 \sum \frac{\mathbf{u}_m^{(n)}}{V_{nm}} \mathbf{M}_{mn} \mathbf{u}_n^{(m)} = \sum \frac{\mathbf{u}_m^{(n)}}{V_{nm}} \mathbf{V}_{mn} \mathbf{u}_n^{(m)}$$

$\omega_n^2 = \text{ratio of 2 positive numbers} > 0$

independent solution: $e^{i\omega t}$

Definition:

$\omega_n \equiv \text{normal mode frequencies}$

$\mathbf{u}_n \equiv \text{normal coordinates}$

$$\eta(t) = 2a_n \mathbf{u}_n e^{i\omega t} + \sum b_n \mathbf{u}_n e^{-i\omega_n t}$$

is the general solution
special properties

If $P_{ij}$ is a positive matrix with eigenvalues $\lambda_i$ and eigenvalues $\rho_i$;

define the matrix $W_{ij} = \lambda_i v^{(i)}$

$\Sigma \rho_m W_{mk} = \lambda_k W_{mk}$

$W_{km} W_{ms} = \delta_{ks}$

$W^T P W = \Lambda I$

$\Lambda$ diagonal matrix

$P = W W^T$

define $P^{+\frac{1}{2}} = W \Lambda^{\frac{1}{2}} W^T$

clearly $P^{+\frac{1}{2}} P^{-\frac{1}{2}} = I$ $P^{+\frac{1}{2}} P^{\frac{1}{2}} = P$

returning to

$(- \omega^2 M + V) \eta_k = 0 \Rightarrow$

$\eta_k = M^{-\frac{1}{2}} \eta^{(k)}$

$(- \omega^2 + M^{-\frac{1}{2}} V M^{-\frac{1}{2}}) M^{-\frac{1}{2}} \eta^{(k)}$

$(- \omega^2 + \tilde{M}^{-\frac{1}{2}} V \tilde{M}^{-\frac{1}{2}}) \tilde{\eta}^{(k)} = 0 \Rightarrow \tilde{\eta}^{(k)} = \tilde{M}^{-\frac{1}{2}} \eta^{(k)}$
This has the form of a standard eigenvalue problem.

The $3^k$ can be chosen to be orthogonal:

$$\vec{\xi}_k \cdot \vec{\xi}_l = S_{k\ell} = \eta^{1/2} \xi_k \eta^{1/2} \xi_l^\top$$

Thus means:

$$\begin{pmatrix} \eta^{(k)} \\ \xi_k \end{pmatrix} \begin{pmatrix} M & C \end{pmatrix} = S_{k\ell}$$

We also have:

$$-\omega_i^2 M \eta^{(k)} = -\nabla \eta^{(k)}$$

$$-\omega_i^2 M \eta^{(k)} = -\eta^{(k)} \omega_i \nabla \eta^{(k)}$$

$$\begin{pmatrix} \vec{\eta}^{(k)} \\ \vec{\nabla} \eta^{(k)} \end{pmatrix} = \omega_i^2 S_{k\ell}$$

These equations assume the $S^{(-1)} = \delta_{k\ell}$.

We see the normal mode eigenvalues simultaneously diagonalize $M$ and $\nabla$. 

\[
\eta^{(k)} = \omega_i^2 \delta_{k\ell}
\]
while a general solution could involve many modes, if the initial condition is in one of the \( \tilde{N}_k \), then the system will oscillate with a fixed normal mode frequency.