Lecture 34

Small amplitude oscillations about stable equilibrium

1. $\tilde{F}_n(q_i, q_k)$

2. Find stable equilibrium

$$\frac{\partial V}{\partial q_k} (\bar{q}_0) = 0 \quad k = 1 \ldots K$$

Condition for equilibrium

Check stability

$$V(\bar{q}) = V(\bar{q}_0) + \sum_{n=1}^{K} \frac{\partial^2 V}{\partial q_n} (\bar{q}_0) (q_n - q_{n0})$$

$$+ \frac{1}{2} \sum_{n=1}^{K} \sum_{m=n+1}^{K} \frac{\partial^2 V}{\partial q_n \partial q_m} (\bar{q}_0) (q_n - q_{n0}) (q_m - q_{m0}) + \ldots$$

The first term is a constant that is unimportant, the second term vanished because $\bar{q}_0$ is an equilibrium point.

The equilibrium is stable if the next term is positive for any non zero $\bar{q} - \bar{q}_0$. 
this term has the form

\[ V(q) = \frac{1}{2} \sum_{m=1}^k V_{mn} (q_m - q_{m0})(q_n - q_{n0}) \]

\[ V_{mn} = \frac{\partial^2 V}{\partial q_m \partial q_n}(q_0) \]

A necessary and sufficient for this correction to be positive is for \( V_{mn} \) to have positive eigenvalues

\[ V^*_{mn} = V_{mn} = V^{**}_{nm} \]

consider the eigenvalue problem

\[ \sum V_{nm} \eta_n = \lambda \eta_m \]

\[ \sum V_{nm} \eta_n = \lambda \eta_n \]

the \( \eta^1, \eta^2 \) be eigenvectors with eigenvalues

\[ \lambda^1, \lambda^2 \]

\[ \sum \eta_n^{(1)} V_{nm} \eta_m^{(1)} = \lambda^{(1)} \sum \eta_n^{(1)} \eta_m^{(1)} \]

\[ \sum \eta_n^{(2)} V_{nm} \eta_m^{(2)} = \lambda^{(2)} \sum \eta_n^{(2)} \eta_m^{(2)} \]

subtract

\[ (\lambda^{(1)} - \lambda^{(2)}) \sum \eta_n^{(2)} \eta_m^{(2)} = 0 \]
It (2) \( \sum \psi_n^+ \psi_n^{(n)} > 0 \) this requires
\( \lambda^{(n)} = \lambda^{(n)*} \) is real.

If \( \lambda^{(n)} \neq \lambda^{(n)} \) then \( \sum \psi_n^+ \psi_n^{(n)} = 0 \).

Since \( \lambda \) and \( \lambda \) are real,

\( \sum \psi_m \eta_m = \lambda \eta_n \)
\( \sum \psi_m \eta_m^* = \lambda \eta_n^* \)

which means that \( \eta_n \) and \( \eta_n^* \) are both eigenvectors \( \Rightarrow \) so is

\( \eta_m = \eta_n \) \( \eta_n^* \)
\( \eta_m^* = i (\eta_n - \eta_n^*) \)

This means that because \( \lambda \) is real,
we can always choose \( \tilde{\eta}^{(r)} \) to be real.

If there are any degeneracies,
we can choose the eigenvectors to be orthogonal,

\( \sum \tilde{\psi}^{(r)} \tilde{\eta}^{(r)} = \lambda^{(r)} \eta^{(r)} \)
\( \tilde{\eta}^{(r)} \tilde{\eta}^{(r)} = \delta_{mn} \)

Since there are \( 1 \) independent eigenvectors - any vector can be expressed as

\( \tilde{\phi} = \sum c_k \tilde{\eta}^{(k)} \)
\[ \sum_{n} \psi_n \psi_m = \]
\[ \sum_{k} \omega_k \psi_n \psi_m \eta_k \]
\[ \sum_{k} \omega_k \lambda^{(k)} s_{nk} = 2 \chi v \lambda^{(k)} \]

The only way for this to be positive for all \( \psi \) is \( \lambda^{(k)} > 0 \).

Conversely, if \( \lambda^{(k)} > 0 \) this is always positive.

Equilibrium is stable if \( V_{nm} = \frac{\partial^2 V}{\partial q_n \partial q_m} \)

has positive eigenvalues.

Small amplitude dynamics

Let \( \eta_n = q_n - q_{nm} \) = displacement from steady equilibrium

\[ \eta_n = \dot{q}_n \]

\[ \Gamma = \sum \frac{1}{2} m \left( \frac{\partial \psi_n}{\partial q_k} \dot{q}_k \cdot \frac{\partial \psi_n}{\partial q_e} \dot{q}_e \right) = \]

\[ = \sum \frac{1}{2} m \left( \frac{\partial \psi_n}{\partial q_k} (\dot{q}_k) \cdot \frac{\partial \psi_n}{\partial q_e} (\dot{q}_e) \right) \dot{\eta}_n \dot{\eta}_n \]
we define the mass \( M_{mnq} \)

\[ M_{mnq}(\vec{q}) = \sum_{k} m_{k} \frac{\partial \tilde{F}_{k}(\vec{q})}{\partial q_{m}} \frac{\partial \tilde{F}_{k}(\vec{q})}{\partial q_{n}} \]

we can expand \( M_{mnq} \) about \( \vec{q}_0 \)

\[ M_{mnq}(\vec{q}) = M_{mnq}(\vec{q}_0) + \sum \frac{\partial M_{mnq}(\vec{q}_0)}{\partial q_{k}} (q_{k} - q_{k}^{(0)}) + \ldots \]

If we assume \( \eta_{k}, \tilde{\eta}_{k} \) are both small then

\[ L = \frac{1}{2} \sum_{m,n} M_{mn}(\vec{q}_0) \eta_{m} \tilde{\eta}_{n} - \frac{1}{2} \sum_{m,n} V_{mn} \eta_{m} \eta_{n} \]

\[ + O(\eta^3) \]

with these approximations \( V_{mn}, M_{mn} \) are constant matrices. Note that

\[ M_{mn} = M_{nm} = M_{mn}^* \]

Lagrangian equations:

\[ \frac{d}{dt} \left( \frac{1}{2} \sum (M_{nk} + M_{kn}) \dot{\eta}_{k} \right) + \frac{1}{2} (V_{nk} + V_{kn}) \eta_{k} = 0 \]

using \( M_{nk} = M_{kn} \) \( V_{nk} = V_{kn} \)
\[ \sum (M_{nk} \eta_k + \nu_n \eta_k) = 0 \]

where \( M \) and \( \nu \) are real positive symmetric matrices.

Try

\[ \eta = \eta(\omega) e^{-i\omega t} \]

Using this in the equations of motion gives

\[ (-\omega^2 M + \nu) \eta = 0 \]

Nonzero solutions require

\[ \det (-\omega^2 M + \nu) = 0 \]

This is a polynomial of degree \( k \) in \( \omega \). It has \( k \) roots \( \omega^2 \) by the fundamental theorem of algebra.

Let \( \omega^2 \eta^{(r)} \) be solutions of this equation.

\[ (-\omega^2 M \eta^{(r)} + \nu \eta^{(r)}) = 0 \]
The $\tilde{m}^n(\omega)$ are called normal mode vectors, $\omega^n_{ik}$ are the corresponding normal mode frequencies.

Next we discuss $\omega^n_{ik}$ $\hat{m}^n_{(k)}$.

* Since $\tilde{M}$ is positive, real, symmetric,

$$\tilde{M} \tilde{\xi}^k = \lambda^k \tilde{\xi}^k, \quad \lambda^k > 0$$

$$\tilde{S}^{\circ} \tilde{M} \tilde{\xi}^k = \lambda^k \tilde{S}^{\circ} \tilde{\xi}^k = \lambda^k S_{ke}$$

$$\tilde{S}^p \tilde{M} \tilde{\xi}^k = \lambda^k \delta_{ke} = \lambda_{ke}$$

Let $O_{nk} = \tilde{\xi}^k_n$

$$\sum O_{nk} O_{mk} = \tilde{S}^{\circ} \tilde{M} \tilde{\xi}^k = S_{ke} = \sum O_{nk}^T O_{nk}$$

$$O_{nk} O_{mk} = \delta_{nk}$$

$$O_{nk}^T O_{nk} = \delta_{nk}$$

$$O_{nk}^T O_{nk} = \lambda_{nk} \delta_{nk}$$

$$\tilde{M} = O \lambda O^T$$

$$\lambda_{nk} = \delta_{mn} \lambda_{nk}$$

$$\lambda \pm \frac{1}{2} = \delta_{mn} \lambda$$

$$\tilde{M}^{\pm \frac{1}{2}} = O \lambda O^T$$

$$\tilde{M}^T = O \lambda O^T$$

$$\tilde{M}^T \tilde{M} = O \lambda^T O \lambda^T = O \lambda O^T = M$$

$$\tilde{M}^T \tilde{M}^T = O \lambda^T O \lambda^T = O I O^T = I$$
\[ \omega_n^2 \overline{M_n} \overline{\eta} = \nabla \overline{\eta} \]
\[ \omega_n \overline{M_n} \overline{\eta} = \nabla M \overline{\eta} \]
\[ \omega \left( M \overline{\eta} \right) = \left( M \overline{\eta} \right) M \overline{\eta} \]

we note \( M \overline{\eta} \overline{\eta} \) is real symmetric and positive

\[ \left( M_{mn} \overline{V_{nk}} M_{k\ell} \right) = \]
\[ M_{k\ell} \overline{V_{nk}} M_{mn} \]
\[ \left( M_{\ell n} \overline{V_{k\ell}} M_{nm} \right) \]

\[ \nabla \cdot \overline{M} \overline{\eta} \nabla \overline{\eta} = \left( \overline{\eta} \right) \left( \overline{\eta} \right) > 0 \]

since \( \nabla > 0 \)

\[ \omega_n^2 \overline{\eta} = \left( \overline{M_n} \right) \overline{\eta} \]

It follows that \( \omega_n > 0 \) \( \omega_n \) real and \( \overline{\eta} \eta_{mn} = \delta_{mn} \)

\[ \therefore \omega_n \text{ real and positive} \]

\[ S_n = \sum_{\ell k} \left( \overline{\eta}_{\ell n} \right) \eta_{\ell k} = M_{\ell k} \overline{\eta}_{\ell n} \eta_{k \ell} \]
\[ = \overline{\eta}_{\ell n} M_{\ell k} \overline{\eta}_{k \ell} \eta_{k \ell} \]
\[ = \overline{\eta}_{\ell n} M_{\ell k} \overline{\eta}_{k \ell} \eta_{k \ell} \]

\[ S_n = \eta_{mn} \overline{\eta}_{n m} \]
\[ \tilde{\eta} \tilde{M} \tilde{\eta} = \delta_{\eta\eta} \]
\[ \tilde{\eta} \tilde{V} \tilde{\eta} = \omega_{\eta} \delta_{\eta\eta} \]

This shows that \( \tilde{\eta}(\cdot) \) diagonalizes both \( \tilde{M} \) and \( \tilde{V} \).

**Example**

\[ \begin{pmatrix} x & m \ y \end{pmatrix} \begin{pmatrix} m & y \end{pmatrix} \]

\[ T = \frac{1}{2} m (x^2 + y^2) \]
\[ V = \frac{1}{2} k \left( x^2 + y^2 + (L - x - y)^2 \right) \]

**Equilibrium**

\[ V = \frac{1}{2} k \left( 2x^2 + 2y^2 + 2xy - 2(x+y)L + L^2 \right) \]
\[ \frac{\partial V}{\partial x} = \frac{1}{2} k \left( 4x + 2y - 2L \right) = 0 \]
\[ \frac{\partial V}{\partial y} = \frac{1}{2} k \left( 4y + 2x - 2L \right) \]

\[ 2x + y = L \]
\[ 2y + x = L \]

*Obvious solution* \( x = y = \frac{L}{3} \)

*Check stability*

\[ \frac{\partial^2 V}{\partial x \partial x} = \begin{pmatrix} 2k & k \\ k & 2k \end{pmatrix} \]
eigenvalues

\[
\begin{pmatrix}
\lambda - 2k & -k \\
-k & \lambda - 2k
\end{pmatrix} - (\lambda - 2k)^2 - k^2 = -
\]

\[(\lambda - 2k)^2 = k^2 \quad \lambda - 2k = \pm k\]

\[\lambda = 2k \pm k = \pm 3k, k\]

both eigenvalues are positive, so the equilibrium is stable.

\[M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\]

equations of motion

\[
\begin{pmatrix}
-\omega^2 m + 2k & k \\
k & -\omega^2 m + 2k
\end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} e^{-i\omega t} = 0
\]

we look for zeros of the determinant

\[(-\omega^2 m + 2k)^2 = k^2\]

\[-\omega^2 m + 2k = \pm k\]

\[-\omega^2 m = -2k \pm k\]

\[\omega^2 = \frac{k}{m} (2 \pm 1) = \frac{k}{m}, \frac{3k}{m}\]
normal modes \( \left( \frac{\kappa}{m} \right) \)

\[
\begin{pmatrix}
-\frac{\kappa}{m} m + 2k & \kappa \\
\kappa & -\frac{\kappa}{m} m + 2k
\end{pmatrix}
= \begin{pmatrix}
\kappa & k \\
k & -\kappa
\end{pmatrix}
\]

\[
\eta = e^{\left( \frac{i}{\kappa} \right)}
\]

\[
\begin{pmatrix}
-\frac{3\kappa}{m} m + 2k & \kappa \\
\kappa & -\frac{3\kappa}{m} m + 2k
\end{pmatrix}
= \begin{pmatrix}
-k & k \\
k & -k
\end{pmatrix}
\]

\[
\eta = e^{\left( \frac{i}{\kappa} \right)}
\]

normalize

\[\eta_i(0) = \frac{1}{\sqrt{m}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \eta_j(0) = \frac{1}{\sqrt{m^2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}\]

\[
\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} = 81i
\]

\[
\frac{1}{\sqrt{2m}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2k & k \\ k & 2k \end{pmatrix} \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} =
\]

\[\frac{1}{2m} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} k \\ -k \end{pmatrix} = \frac{k}{m}
\]

\[
\frac{1}{\sqrt{2m}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2k & k \\ k & 2k \end{pmatrix} \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} =
\]

\[\frac{1}{2m} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3k \\ -k \end{pmatrix} = \frac{3k}{m}
\]
\[
\frac{1}{V_{2m}} (1-1) \left( \begin{array}{c}
2k \\
R \\
2k 
\end{array} \right) \frac{1}{V_{2m}} (1) = \\
\frac{1}{V_{2m}} (1-1) \frac{1}{V_{2m}} \left( \begin{array}{c}
3k \\
3k 
\end{array} \right) = 0
\]

The modes have the structure

\[
\begin{array}{c}
\vdots \\
1 \\
\vdots \\
\end{array} \quad \begin{array}{c}
\vdots \\
1 \\
\vdots \\
\end{array} \quad \begin{array}{c}
\vdots \\
1 \\
\vdots \\
\end{array} \quad \begin{array}{c}
\vdots \\
1 \\
\vdots \\
\end{array}
\end{array}
\]
\[
M = \frac{1}{V_{2m}} (1) \quad \omega = \frac{3k}{\omega_n}
\]
\[
L = \frac{1}{V_{2m}} (1) \quad \omega = \frac{k}{\omega_m}
\]

**Hamiltonian Dynamics**

**Legendre transformation**

1. Define \( P_k = \frac{\partial L}{\partial \dot{q}_k} \)
2. \( H(p, q) = \sum P_k \dot{q}_k - L(q, \dot{q}) \)

\[
\frac{\partial H}{\partial \dot{q}_n} = P_n - \frac{\partial L}{\partial q_n} = 0 \quad \Rightarrow 0 = L
\]

\( H(p, q) \)
\[
\frac{\partial H}{\partial p_k} = q_k \\
\frac{\partial H}{\partial q_k} = -\frac{\partial L}{\partial p_k} = -\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) = -\frac{d}{dt} p_k
\]

Use Lagrange's equations:

\[
\frac{\partial L}{\partial q_k} = -\frac{\partial H}{\partial p_k} = -\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) = -\frac{d}{dt} p_k
\]

These give:

\[
\begin{align*}
\dot{q}_k &= \frac{\partial H}{\partial p_k} \\
\dot{p}_k &= -\frac{\partial H}{\partial q_k}
\end{align*}
\]

These are called Hamilton's equations (clearly they follow from Lagrangian equations).

Consider:

\[
L' = \sum p_k \dot{q}_k - H
\]

\[
\frac{\partial L'}{\partial p_k} = q_k - \frac{\partial H}{\partial p_k} = 0
\]

gives \( q_k \) in terms of \( p_k \).

\[
\frac{\partial L'}{\partial q_k} = p_k - \frac{\partial H}{\partial q_k} = +\dot{p}_k
\]

\[
= d \left( \frac{\partial L'}{\partial \dot{q}_k} \right)
\]

\[
\frac{d}{dt} \left( \frac{\partial L'}{\partial \dot{q}_k} \right) - \frac{\partial L'}{\partial q_k} = 0
\]
This shows that using the Lagrange transforation on $H$ recover Langrange equations.

The condition for this to work is

$$\frac{\partial^2 L}{\partial q_i \partial q_j} = \left( \frac{\partial p_j}{\partial q_i} \right) \neq 0 \quad \text{(never 0)}$$

This means that it is possible to solve for $p_k$ as a function of the $q_k$ (this result is called the inverse function theorem).

Example:

$$L = \frac{1}{2} m x^2 - V(x)$$

$$p = \frac{\partial L}{\partial x} = m \dot{x} \quad x = \frac{p}{m}$$

$$H = p \dot{x} - \frac{1}{2} m \dot{x}^2 + V(x)$$

$$= \frac{p^2}{2m} - \frac{1}{2} m \left( \frac{p^2}{2m} \right) + V(x)$$

$$= \frac{p^2}{2m} + V(x)$$
\[
\dot{x} = \frac{\partial H}{\partial \dot{p}} = \frac{p}{m}
\]

\[
\dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x} = mx
\]

This recovers both Lagrange's equation and Newton's second law.