Hamiltonian Mechanics  HW- ch12 1.4

1. Start with Lagrangian
2. Construct generalized momenta

\[ p_n = \frac{\partial L}{\partial \dot{q}_n} \]

\[ \frac{\partial^2 L}{\partial q_n \partial \dot{q}_m} > 0 \] means that it is possible to solve for \( q_n \) in terms of \( p_n \) uniquely

(The relevant theorem is the inverse function theorem)

3. Perform the Legendre transformation

\[ H(q, \dot{q}) = \sum p_n \dot{q}_n - L(q, \dot{q}, t) \]

\[ \frac{\partial H}{\partial \dot{q}} = p_n - \frac{\partial L}{\partial q_n} = 0 \]

\[ \frac{\partial H}{\partial p_n} = \dot{q}_n \]

\[ \frac{\partial H}{\partial q} = - \frac{\partial L}{\partial q_n} \]

4. Use Lagrange's equations

\[ \frac{\partial H}{\partial q_n} = - \frac{\partial L}{\partial q_n} = - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_n} \right) = - \frac{d}{dt} p_n \]

(replace \( \dot{q}_n \) by \( \dot{q}_n (\bar{q}, \bar{\dot{q}}) \))
The resulting equations are called Hamilton's equations:

\[ \dot{q}_n = \frac{\partial H}{\partial p_n} \]
\[ \dot{p}_n = -\frac{\partial H}{\partial q_n} \]

This is a system of \( 2N \) first order differential equations.

Examples:

\[ L = \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\theta}^2 \right) - V(r) \]

\[ p_r = \frac{\partial L}{\partial \dot{r}} = m \dot{r} \]
\[ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \]

Solve \( r, \theta \) in terms of \( p_r, p_\theta \)

\[ \dot{r} = \frac{p_r}{m} \]
\[ \dot{\theta} = \frac{p_\theta}{m r^2} \]

Legendre transformation

\[ H(p_r, p_\theta, r) = \dot{r} p_r + \dot{\theta} p_\theta - \left( \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - V(r) \right) \]

Replace \( r, \theta \)

\[ H(p_r, p_\theta, r) = \frac{p_r^2}{m} + \frac{p_\theta^2}{m r^2} - \frac{1}{2} m \left( \frac{p_r}{m} \right)^2 - \frac{1}{2} m r^2 \left( \frac{p_\theta}{m r^2} \right)^2 + V(r) \]

\[ = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2m r^2} + V(r) \]
Hamilton's equations
\[ \dot{r} = \frac{\partial H}{\partial \dot{r}} = \frac{P_r}{m} \]
\[ \dot{r}_r = -\frac{\partial H}{\partial \dot{r}_r} = -2 \frac{P_r^2}{2m r^3} - \frac{2V}{r} \]
\[ \dot{\theta} = \frac{\partial H}{\partial \dot{\theta}} = \frac{P_\theta}{2m r^2} \]
\[ \dot{P}_r = 0 \]

Since \( H \) is independent of \( \theta \), \( P_\theta \) is a conserved quantity.
(\( \theta \) is called an ignorable coordinate.)

If we differentiate \( \dot{r} \)
\[ \ddot{r} = \frac{\dot{P}_r}{m} = \frac{1}{m} \left( -\frac{\partial H}{\partial \dot{r}_r} \right) = -\frac{P_r^2}{m r^3} - \frac{2V}{r} \]

This is a second order equation for \( r \).

We can compare this to Lagrange's equations:
\[ \frac{\partial L}{\partial \dot{r}} = m \dot{r} \quad \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \]
\[ \frac{\partial L}{\partial r} = m r \ddot{\theta} - \frac{\partial V}{\partial r} - \frac{\partial L}{\partial \theta} = 0 \]
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \dot{\theta} = 0 \Rightarrow m r^2 \ddot{\theta} = \text{const} \]
\[ \frac{d}{dt} (m \dot{r}) = m \ddot{r} - m r \dot{\theta}^2 + \frac{\partial V}{\partial r} \]
\[ m \ddot{r} = mr \left( \frac{c^2}{m^2 r^4} \right) - \frac{2V}{r} \]

\[ \ddot{r} = \frac{c^2}{m^2 r^3} - \frac{2V}{mr} \]

We see that we get the same equation from Lagrange's equations where the constant \( c \) is the conserved \( P_0 \).

**Example 2: Particle in EM field**

\[ L = \frac{1}{2} m \dot{r}^2 + q \dot{r} \cdot A(r,t) - q \phi(r,t) \]

Find the generalized momentum

\[ \vec{p}_r = \frac{\partial L}{\partial \dot{r}} = m \dot{r} + q \vec{A}(r,t) \]

In this case, the generalized momentum is not the ordinary momentum

\[ \vec{p} = \frac{\vec{p}_r}{m} - \frac{q}{m} \vec{A} \]

\[ H = \vec{p}_r \cdot \vec{r} - L \]

\[ = \vec{p}_r \cdot \vec{r} - \frac{1}{2} m \dot{r}^2 - q \dot{r} \cdot A + q \phi(r,t) \]

\[ = \vec{p}_r \left( \frac{\vec{p}_r}{m} - \frac{q}{m} \vec{A} \right) - \frac{1}{2} m \left( \frac{\vec{p}_r}{m} - \frac{q}{m} \vec{A} \right) \]

\[ - q \left( \frac{\vec{p}_r}{m} - \frac{q}{m} \vec{A} \right) \cdot \vec{A} + q \phi \]
\[
\frac{P_r^2}{3l} - \frac{9}{8} \frac{P_r \cdot \vec{A}}{m} - \frac{1}{2} \frac{P_r^3}{m^2} + \frac{9}{m} \frac{P_r' \cdot \vec{A}}{m} - \frac{a^2}{2m} A^2 \\
- \frac{a}{m} \frac{P_r \cdot \vec{A}}{m} + \frac{a^2}{m^2} A^2 + q \Phi \\
= \frac{P_r^2}{2m} - \frac{a}{m} \frac{P_r' \cdot \vec{A}}{m} + \frac{a^2}{2m^2} A^2 + q \Phi
\]

Conservation Laws - Poisson Brackets

Let \( F (\vec{p}, \vec{q}, t) \) be any function of \( \vec{p}, \vec{q}, t \)

\[
\frac{dF}{dt} = \frac{\partial F}{\partial \vec{p}} \cdot \vec{p}' + \frac{\partial F}{\partial \vec{q}} \cdot \vec{q}' + \frac{\partial F}{\partial t}
\]

Using Hamilton equations

\[
\frac{dF}{dt} = \frac{\partial}{\partial \vec{p}} \left( \vec{p} \left( -\frac{\partial H}{\partial \vec{q}} \right) + \vec{q} \left( \frac{\partial H}{\partial \vec{p}} \right) \right) + \frac{\partial F}{\partial t}
\]

\[
\frac{dF}{dt} = \frac{\partial}{\partial \vec{q}} \left( \frac{\partial F}{\partial \vec{p}} \frac{\partial H}{\partial \vec{q}} - \frac{\partial F}{\partial \vec{q}} \frac{\partial H}{\partial \vec{p}} \right) + \frac{\partial F}{\partial t}
\]

The quantity

\[
\{ F, H \} = \frac{\partial}{\partial \vec{p}} \left( \frac{\partial F}{\partial \vec{q}} \frac{\partial H}{\partial \vec{q}} - \frac{\partial F}{\partial \vec{q}} \frac{\partial H}{\partial \vec{p}} \right)
\]

is called the Poisson bracket of \( F \) with \( H \).
If \( F(q,p) \) has no explicit time dependence then

\[
\frac{dF}{dt} = \{ F, H \}
\]

If the Poisson bracket vanishes then \( F \) is a conserved quantity.

Properties of the Poisson Bracket:

1. \( \{ F, G \} = -\{ G, F \} \)
2. \( \{ A, BC \} = \{ AB, C \} + B \{ A, C \} \)
3. \( \{ A, B + c C \} = \{ A, B \} + c \{ A, C \} \) \( c = \text{constant} \)
4. \( \{ A \{ B, \{ C, \{ D \} \} \} + \{ B \{ C, \{ D \} \}, A \} + 3 C \{ A \{ B, C \}, A \} = 3 C E \{ A B, D \} \) (Jacobi Identity)

Remarks

If \( H \) has no explicit time dependence then

\[
\frac{dH}{dt} = \{ H, H \} = -\{ H, H \} = 0
\]

This means that \( H \) is a conserved quantity whenever it has no explicit time dependence.
Assume $F, G$ are conserved.

\[ \{ F H^3 \} = \{ G H^3 \} = 0 \]

\[ \{ \{ F G \} H^3 \} = -\{ H \{ F G \} H \} = \]

\[ = \{ F \{ G H^3 \} \} + \{ G \{ H F^3 \} \} = \]

\[ = \{ F \{ G H^3 \} \} - \{ G \{ F H^3 \} \} \]

This means that if $F$ and $G$ are conserved then $\{ F G \}$ is conserved.

Note: $\omega$ are easy to show.

2) $\{ A, B, C \} =$

\[ = \sum \left( \frac{\partial A}{\partial n} \frac{\partial (BC)}{\partial p_n} - \frac{\partial A}{\partial p_n} \frac{\partial (BC)}{\partial q_n} \right) = \]

\[ = \sum \left( \frac{\partial A}{\partial q_n} \left( \frac{\partial B}{\partial p_n} C + B \frac{\partial C}{\partial p_n} \right) \right) = \]

\[ = \sum \left( \frac{\partial A}{\partial q_n} \frac{\partial B}{\partial p_n} - \frac{\partial A}{\partial p_n} \frac{\partial B}{\partial q_n} \right) + B \sum \left( \frac{\partial A}{\partial q_n} \frac{\partial C}{\partial p_n} - \frac{\partial A}{\partial p_n} \frac{\partial C}{\partial q_n} \right) = \]

\[ C \{ A B C \} + B \{ A C \} \]

This is just a consequence of the chain rule.
\[ \mathcal{E} \mathcal{A} \mathcal{B} \mathcal{C} \mathcal{A} \mathcal{B} + \mathcal{E} \mathcal{B} \mathcal{C} \mathcal{A} \mathcal{B} \mathcal{A} + \mathcal{C} \mathcal{E} \mathcal{A} \mathcal{B} \mathcal{A} = 0 \]

Note:

\[ \mathcal{E} \mathcal{A} \mathcal{E} \mathcal{B} \mathcal{C} \mathcal{A} \mathcal{B} = \sum \frac{\partial A}{\partial a_n} \frac{\partial C}{\partial a_m} \left( \frac{\partial B}{\partial p_m} \frac{\partial C}{\partial p_n} - \frac{\partial B}{\partial p_n} \frac{\partial C}{\partial p_m} \right) - \frac{\partial A}{\partial p_n} \frac{\partial C}{\partial a_n} \left( \frac{\partial B}{\partial p_m} \frac{\partial C}{\partial p_m} - \frac{\partial B}{\partial p_m} \frac{\partial C}{\partial p_m} \right) \]

Looking at this we see that the sum of products of 2 terms with single derivatives and one term with 2 derivatives.

To show the cancelation we focus on the terms where \( \mathcal{C} \) has 2 derivatives – these come from \( \mathcal{E} \mathcal{A} \mathcal{B} \mathcal{C} \mathcal{A} \mathcal{B} \mathcal{C} \) as

\[ \mathcal{E} \mathcal{B} \mathcal{C} \mathcal{A} \mathcal{B} \mathcal{A} \mathcal{C} \mathcal{E} = \sum \frac{\partial A}{\partial a_n} \frac{\partial C}{\partial a_m} \left( \frac{\partial B}{\partial p_m} \frac{\partial C}{\partial p_n} - \frac{\partial B}{\partial p_n} \frac{\partial C}{\partial p_m} \right) - \frac{\partial A}{\partial p_n} \frac{\partial C}{\partial a_n} \left( \frac{\partial B}{\partial p_m} \frac{\partial C}{\partial p_m} - \frac{\partial B}{\partial p_m} \frac{\partial C}{\partial p_m} \right) \]

This shows that these terms all vanish.
By symmetry - the second derivative terms in A and B all cancel. We can write this as

\[ \{ A \{ B C \} \} = -\{ B \{ C A \} \} - \{ C \{ E A B \} \} = \{ B \{ E A C \} \} + \{ \{ A B \} , C \} \]

This looks something like the chain rule.

Remarks

\[ \frac{dP_n}{dt} = \{ P_n H \} = \sum \left( \frac{\partial P_n}{\partial \rho_m} \frac{\partial H}{\partial \rho_m} - \frac{\partial P_n}{\partial \rho_m} \frac{\partial H}{\partial \rho_m} \right) \]

\[ = -\frac{\partial H}{\partial \rho_n} \]

\[ \frac{d\rho_n}{dt} = \{ \rho_n H \} = \sum \left( \frac{\partial \rho_n}{\partial \rho_m} \frac{\partial H}{\partial \rho_m} - \frac{\partial \rho_n}{\partial \rho_m} \frac{\partial H}{\partial \rho_m} \right) \]

\[ = \frac{\partial H}{\partial \rho_{m_0}} \]

There are Hamilton's equations:

\[ \{ q_n p_m \} = \sum \left( \frac{\partial q_n}{\partial \rho_m} \frac{\partial p_m}{\partial \rho_m} - \frac{\partial q_n}{\partial \rho_m} \frac{\partial p_m}{\partial \rho_m} \right) \]

\[ = \delta_{nm} \]

\[ \sum q_n p_m = \delta_{nm} \]
Note \( \ln F = F(q, \dot{q}) \)

\[
\frac{dF}{dt} = \{F, H\}
\]

\( \{F, H\} \) is a new function of \( \ddot{q} \dot{p} \)

\[
\frac{d^2F}{dt^2} = \{\{F, H\}, H\} = \{\{F, H\}, H\}
\]

\[
\frac{d^3F}{dt^3} = \{\{\{F, H\}, H\}, H\}
\]

\[
\frac{d^NF}{dt^N} = \{\{\ldots \{F, H\}, H\}, \ldots, H\} \quad \text{N times}
\]

If the Taylor series converges

\[
F(p(t), q(t)) = F(\tilde{p}(0), \tilde{q}(0)) + \sum_{n=1}^{\infty} \frac{t^n}{n!} \frac{d^n}{dt^n} (F(\dot{q}, \ddot{p}))_{t=0}
\]

we use the notation

\[
D_H A = \{A, H\}
\]

\[
F(p(t), q(t)) = \sum_{n=0}^{\infty} \frac{t^n}{n!} D_H^n F(q(0), p(0))
\]

\[
= e^{t D_H} F(q(0), p(0))
\]
\[
\begin{align*}
\tilde{p}(t) &= e^{\tilde{D} t} p(0) \\
\tilde{q}(t) &= e^{\tilde{D} t} q(0)
\end{align*}
\]

**Example: Harmonic Oscillator**

\[
L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2
\]

\[
P = \frac{2L}{\partial x} = m \dot{x} \quad x = \frac{P}{m}
\]

\[
H = p \dot{x} - L = \frac{p^2}{2m} - \frac{P^2}{2m} + \frac{1}{2} k x^2
\]

\[
= \frac{p^2}{2m} + \frac{1}{2} k x^2
\]

Since this has no explicit time dependence, \( H \) is conserved — it is just the energy of the oscillator.

\[
\begin{align*}
\tilde{D} H P &= \{ \tilde{H}, \{ \tilde{P}, \frac{P^2}{2m} + \frac{1}{2} k x^2 \} \} = \\
&= \frac{1}{2m} (P \{ \tilde{P}, \tilde{P} \tilde{P} \tilde{P} \tilde{P} \} + \frac{1}{2} k \{ \tilde{P}, \tilde{P} \tilde{P} \tilde{P} \tilde{P} \}) \\
&= 0
\end{align*}
\]

\[
\begin{align*}
\tilde{D} H \dot{P} &= -k \dot{P} \\
\tilde{D} H \dot{X} &= \{ \tilde{H}, \{ \tilde{X}, \frac{P^2}{2m} + \frac{1}{2} k x^2 \} \} = \\
&= \frac{1}{2m} (\{ \tilde{X}, \tilde{P} \tilde{P} \tilde{P} \tilde{P} \} + \frac{k}{2} \{ \tilde{X}, \tilde{P} \tilde{P} \tilde{P} \tilde{P} \}) \\
&= \frac{P}{m}
\end{align*}
\]
\[ D_2^2 X = D_H \frac{p}{m} = \frac{1}{m} (-kx) = -\frac{k}{m} x \]

\[ D_H^{2n} X = (-\frac{k}{m})^n x \]

\[ D_H^{2n+1} X = (-\frac{k}{m})^n D_H x = \frac{1}{m} (-\frac{k}{m})^n p \]

\[ X(t) = \sum \frac{t^n}{n!} D_H^n x = \]

\[ = \sum \frac{t^n}{(2n)!} (-\frac{k}{m})^n x + \frac{1}{m} \sum \frac{t^{2n+1}}{(2n+1)!} \left(-\frac{k}{m}\right)^n p \]

\[ = \sum \frac{(-1)^n}{(2n)!} \left(\frac{1}{\sqrt{m}}t\right)^{2n} \frac{1}{m} \sqrt{\frac{m}{k}} \sum \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{\sqrt{m}}t\right)^{2n+1} D_\theta \]

\[ X(t) = X(0) \cos \left(\sqrt{\frac{k}{m}} t\right) + \frac{p}{\sqrt{mk}} \sin \left(\sqrt{\frac{k}{m}} t\right) \]

\[ p(t) = m \frac{dx}{dt} = \]

\[ = m \sqrt{\frac{k}{m}} x(0) \sin \left(\sqrt{\frac{k}{m}} t\right) + p(0) \cos \left(\sqrt{\frac{k}{m}} t\right) \]

These are the well-known solutions to the harmonic oscillator equation. In this case, \( p, q \)

are functions of the initial conditions.
canonica quantization

\[ \dot{p}_k = \{ p_k, H \} \]
\[ \dot{q}_k = \{ q_k, H \} \]
\[ \{ q_k, p_{\ell} \} = i \hbar \delta_{k\ell} \]

In quantum mechanics, \( p_k, q_k \) become operators

\[ p_k = \frac{\hbar}{i} \frac{\partial}{\partial q_k} \]
\[ H = \frac{p_k^2}{2m} + V(q) \]
\[ = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial q_k^2} + V(q) \]

\[ \{ q_k, p_n \} - \{ p_n, q_k \} = \frac{\hbar}{i} \left( q_k \frac{\partial}{\partial q_n} - q_n \frac{\partial}{\partial q_k} \right) \]
\[ = \frac{\hbar}{i} \delta_{nk} + \frac{\hbar}{i} \left( q_k \frac{\partial}{\partial q_n} - q_n \frac{\partial}{\partial q_k} \right) \]
\[ = \frac{\hbar}{i} \delta_{nk} \]
\[ \left[ q_k, p_n \right] = \frac{\hbar}{i} \delta_{nk} = \frac{\hbar}{i} \left[ q_{\ell k}, p_n \right] \]

\[ \left[ q_k, H \right] = \sum \left[ q_k, \frac{p_k'}{2m} \right] + \left[ q_{\ell k}, V \right] \]
\[ = -\sum \left[ -\hbar^2 \frac{\partial^2}{\partial q_k}, q_n \right] \frac{1}{c m} \]
\[ = \frac{\hbar^2}{m} \sum \delta_{nk} \frac{2}{\partial q_k} \]
\[ = -i \frac{\hbar}{m} \left( \frac{\hbar}{i} \frac{2}{\partial q_k} \right) = -i \frac{\hbar}{m} p_{\ell k} \]
\[ \frac{p_k}{\hbar} = \frac{1}{i} \frac{\partial}{\partial x_k} = \frac{1}{i} \{q_k, H \} \]

This suggests the rule:

\[ \{q_B, H \} \to \frac{\hbar}{i} \{q_B, H \} \]

as a rule for passing from classical to quantum mechanics—this is called canonical quantization.