Last time

\[ L = T - V \]

\[ p_n = \frac{\partial L}{\partial q_n} = 0 \quad \dot{q}_n = \dot{q}_n (\ddot{p}, \ddot{q}) \]

\[ H = \sum p_n \dot{q}_n - L \]

In most cases

\[ H = T + V = \text{total energy} \]

Equations of motion

\[ \dot{q}_n = \frac{\partial H}{\partial \dot{p}_n} \]

\[ \ddot{p}_n = -\frac{\partial H}{\partial q_n} \]

Formal solution method

\[ q_n (t) = q_n (t_0) + \int_{t_0}^{t} \frac{\partial H}{\partial \dot{p}_n} (q(n), p(t')) dt' \]

\[ p_n (t) = p_n (t_0) - \int_{t_0}^{t} \frac{\partial H}{\partial q_n} (q(n), p(t')) dt' \]

For small \( t-t_0 \), the solution by iteration converges

\[ q_n^{(0)} (t) = q_n (t_0) \]

\[ p_n^{(0)} (t) = p_n (t_0) \]

\[ q_n^{(k)} (t) = q_n (t_0) + \int_{t_0}^{t} \frac{\partial H}{\partial \dot{p}_n} (q^{(k-1)} (t'), p^{(k-1)} (t')) dt' \]

\[ p_n^{(k)} (t) = p_n (t_0) - \int_{t_0}^{t} \frac{\partial H}{\partial q_n} (q^{(k-1)} (t'), p^{(k-1)} (t')) dt' \]

\[ q_n (t) = \lim_{k \to \infty} q_n^{(k)} (t) \]

\[ p_n (t) = \lim_{k \to \infty} p_n^{(k)} (t) \]
\( F(qH, pH, t) \)

\[
\frac{dF}{dt} = \{ F, H \} + \frac{\partial F}{\partial t}
\]

\( \{ F, G \} \) is called a Poisson bracket

\[
\{ F, G \} = \sum \left( \frac{\partial F}{\partial q_k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q_k} \right)
\]

The Poisson bracket has the following properties:

1. \( \{ F, G \} = -\{ G, F \} \)
2. \( \{ F, G + cH \} = \{ F, G \} + c \{ F, H \} \)
   
   \( c = \text{constant} \)
   
   \( F, G, H \) functions of \( p, q, t \)
3. \( \{ F, GH \} = \{ F, G \} H + \{ F, H \} G \)
4. \( \{ F \{ G \{ H \} \} \} = \{ G \{ E \{ H \} \} \} + \{ H \{ E \{ G \} \} \}
   
   \( \text{Jacobi identity} \)

Remarks:

5. If \( H \) has no explicit time dependence then

\[
\frac{dH}{dt} = \{ H, H \} = -\{ H, H \} = 0
\]

This means that \( H \) is a conserved quantity.
If \( G \) has no explicit time dependence and
\[
\{E, H \} = 0
\]
then \( G \) is a constant of motion.

If \( F, \tilde{F} \) are constants of motion, then so is \( \{F, \tilde{F}\} \)

Note
\[
\begin{align*}
\dot{\rho}^\mu &= \{P, H\}^\mu = \frac{2}{\pi} \left( \frac{\Theta^\rho \Theta^H}{\Theta^{\mu \rho} \Theta^m} - \frac{\Theta^\rho \Theta^H}{\Theta^{\mu \rho} \Theta^m} \right) \\
&= \frac{2}{\pi} \left( -\delta^{\mu \rho} \frac{\partial H}{\partial \rho} \right) \\
&= -\frac{Q H}{\Theta^{\rho \mu}} \\
\dot{\eta}^\mu &= \{\eta, H\}^\mu = \frac{2}{\pi} \left( \frac{\Theta^\rho \Theta^H}{\Theta^{\mu \rho} \Theta^m} - \frac{\Theta^\rho \Theta^H}{\Theta^{\mu \rho} \Theta^m} \right) \\
&= \frac{2}{\pi} \delta^{\rho \mu} \frac{Q H}{\Theta^{\rho \mu}} \\
&= \frac{Q H}{\Theta^{\rho \mu}}
\end{align*}
\]

If \( F = F(\rho, \tilde{\rho}) \)
\[
\frac{dF}{dt} = \{F, H\}
\]
but \( \{F, H\} \) is another function of \( \rho \) and \( \tilde{\rho} \).
\[
\frac{d^2 F}{dt^2} = \frac{d}{dt} \{ F, H \} = \{ F, \{ H, H \} \}
\]

\[
\frac{d^n F}{dt^n} = \{ \cdots \{ F, H \} \{ H \} \cdots \} = H
\]

we introduce the short hand notation

\[
D_H F = \{ F, H \}
\]

\[
\frac{d^n F}{dt^n} = D^n_H F
\]

If the Taylor series for \( F(\tilde{q}(t), \tilde{p}(t)) \) vanishes then

\[
F(\tilde{q}(t), \tilde{p}(t)) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{d^n F}{dt^n} (\tilde{q}(t), \tilde{p}(t)) = \sum_{n=0}^{\infty} \frac{t^n}{n!} D^n_H F(\tilde{q}(t), \tilde{p}(t))
\]

case - Harmonic Oscillator

\[
H = \frac{p^2}{2m} + \frac{1}{2} k x^2
\]

\[
\{ x, H \} = \frac{1}{2m} \{ x, \{ p^2 \} \} + \frac{1}{2} k \{ x, x^2 \}
\]

\[
= \frac{1}{2m} (p \{ x, p \} + \{ x, p \} p) + \frac{k}{2} (x \{ xx \} + \{ xx \} x)
\]
\[ \mathbf{r} \times \mathbf{p} = \left( \frac{\partial}{\partial r} \mathbf{r} \cdot \mathbf{p} - \mathbf{p} \cdot \frac{\partial}{\partial r} \mathbf{r} \right) = 1 \]
\[ \mathbf{r} \times \mathbf{r} = -\mathbf{r} \times \mathbf{r} = 0 \]

It follows that
\[ \mathbf{r} \times \mathbf{H} = \mathbf{D} \mathbf{r} \mathbf{x} = \frac{\mathbf{p}}{2m} = \frac{\mathbf{p}}{m} \]

Similarly
\[ \mathbf{r} \times \mathbf{H} = \mathbf{D} \mathbf{r} \mathbf{p} = \mathbf{r} \left[ \frac{\mathbf{p}}{2m} \mathbf{p}^2 + \frac{1}{2} \mathbf{k} \mathbf{r} \times \mathbf{r} \right] \]
\[ = \frac{1}{2m} \left( \mathbf{p} \mathbf{p} \mathbf{p} \mathbf{p} + \mathbf{p} \mathbf{p} \mathbf{p} \mathbf{p} \right) + \frac{1}{2} \mathbf{k} \left( \mathbf{r} \mathbf{r} \mathbf{r} \mathbf{r} \mathbf{r} \mathbf{r} \mathbf{r} \mathbf{r} \mathbf{r} \mathbf{r} \mathbf{r} \right) \]
\[ = -\mathbf{k} \mathbf{r} \]

Using these identities:
\[ \mathbf{D} \mathbf{r} \mathbf{x} = \frac{\mathbf{p}}{m} \]
\[ \mathbf{D} \mathbf{r} \mathbf{p} = \frac{1}{m} \mathbf{D} \mathbf{r} \mathbf{p} = -\frac{\mathbf{k}}{m} \mathbf{r} \]
\[ \mathbf{D} \mathbf{r} \mathbf{x} = -\frac{\mathbf{k}}{m} \mathbf{D} \mathbf{r} \mathbf{x} = -\frac{\mathbf{k}}{m} \frac{\mathbf{p}}{m} \]
\[ \mathbf{D} \mathbf{r} \mathbf{x} = -\frac{\mathbf{k}}{m} \frac{1}{m} (-\mathbf{k} \mathbf{r}) = \frac{\mathbf{k}^2 \mathbf{r}}{m^2} \]

Similarly,
\[ \mathbf{D} \mathbf{r} \mathbf{p} = -\mathbf{k} \mathbf{r} \]
\[ \mathbf{D} \mathbf{r} \mathbf{p} = -\mathbf{k} \mathbf{D} \mathbf{r} \mathbf{x} = -\frac{\mathbf{k}}{m} \mathbf{p} \]
\[ \mathbf{D} \mathbf{r} \mathbf{p} = -\frac{\mathbf{k}}{m} \mathbf{D} \mathbf{r} \mathbf{p} = -\frac{\mathbf{k}}{m} (-\mathbf{k} \mathbf{r}) \]
\[ \mathbf{D} \mathbf{r} \mathbf{p} = -\frac{\mathbf{k}^2 \mathbf{r}}{m} \frac{\mathbf{p}}{m} = \frac{\mathbf{k}^2 \mathbf{r}}{m^2} \mathbf{p} \]

\[ \vdots \]
Putting everything together

\[ x(t) = \sum \frac{t^n}{n!} D^n x(0) \]
\[ = \sum \frac{t^{2n}}{2n!} D^{2n} x(0) + \sum \frac{t^{2n+1}}{(2n+1)!} D^{2n+1} x(0) \]
\[ = \sum \frac{t^{2n}}{(2n)!} (-\frac{R}{m})^n x(0) + \sum \frac{t^{2n+1}}{(2n+1)!} (-\frac{R}{m})^n \frac{p(0)}{m} \]
\[ = \sum \frac{t^{2n}}{(2n)!} \left( \frac{\sqrt{R}}{m} \right)^n x(0) + \sum \frac{t^{2n+1}}{(2n+1)!} \left( \frac{\sqrt{R}}{m} \right)^{2n+1} \frac{m}{\sqrt{R}} \frac{p(0)}{m} \]

\[ x(t) = \cos \left( \frac{\sqrt{R}}{m} t \right) x(0) + \frac{1}{\sqrt{mR}} \sin \left( \frac{\sqrt{R}}{m} t \right) p(0) \]

\[ p(t) = m \dot{x}(t) = \]
\[ = m \left( \frac{\sqrt{R}}{m} \sin \left( \frac{\sqrt{R}}{m} t \right) x(0) + \sqrt{mR} \frac{\sqrt{R}}{m} \cos \left( \frac{\sqrt{R}}{m} t \right) p(0) \right) \]
\[ = -\sqrt{mR} \sin \left( \frac{\sqrt{R}}{m} t \right) x(0) + \cos \left( \frac{\sqrt{R}}{m} t \right) p(0) \]

We see this gives exactly the time-dependent solutions as a function of initial conditions

\[ x(t) = e^{tD_H} x(0) \]
\[ p(t) = e^{tD_H} p(0) \]
canonicaL quantization

Note

\[ \epsilon q_n p_m = \frac{2}{k} \left( \frac{\partial q_n}{\partial q_k} \frac{\partial p_m}{\partial p_k} - \frac{\partial q_n}{\partial p_k} \frac{\partial p_m}{\partial q_k} \right) \]

\[ = \frac{2}{k} \delta_{nk} \delta_{km} \]

\[ = \delta_{nm} \]

In quantum mechanics, \( p_n, q_n \) become operators

\( p_n = -i\hbar \frac{\partial}{\partial q_n} \)

\[ [q_n, p_m] = q_n p_m - p_m q_n = \]

\[ = q_n (-i\hbar \frac{\partial}{\partial q_m}) - (-i\hbar \frac{\partial}{\partial q_m}) q_n = \]

\[ = -i\hbar q_n \frac{\partial}{\partial q_m} + i\hbar \delta_{mn} + i\hbar q_n \frac{\partial}{\partial q_m} \]

\[ = i\hbar \delta_{mn} \]

Comparing these expressions

\[ [q_n, p_m] = i\hbar [\delta_{nm}, p_m] \]

The procedure \( q_n p_m \rightarrow \) operators and commutator \( \rightarrow \) replace by \( i\hbar \times \) poisson brackets is called canonical quantization.
Consider

\[ H = \sum \frac{p_i^2}{2m} + V(q_i, q_n) \]

\[ [H, P_m] = \sum \left[ \frac{p_i^2}{2m}, P_m \right] + [V, P_m] \]

\[ = \frac{1}{2m} \sum \left[ \frac{p_i^2}{2m}, P_m \right] \quad \text{vanishes} \]

\[ [V, P_m] = V(-i\hbar \frac{\partial}{\partial q_m}) - (-i\hbar \frac{\partial}{\partial q_m}) V \]

\[ = i\hbar \frac{\partial V}{\partial q_m} \]

\[ = i\hbar \{V, P_m\} = i\hbar \{H, P_m\} \]

\[ \therefore [H, P_m] = i\hbar \{H, P_m\} \]

We see that this prescription is consistent.

What is different

\[ q_p q = q^3 p \quad \text{in quantum mechanics} \]

\[ q_p q = q^3 p = pq^3 \quad \text{in classical mechanics} \]

Canonical quantization

1. Write the classical Hamiltonian
2. Replace \( q_n, p_n \) by operators
3. Replace commutator by \( i\hbar \{, \} \)
canonical transformations
\[ \bar{q} \bar{p} \to \bar{q}'(\bar{q}, \bar{p}) \bar{p}'(\bar{q}, \bar{p}) \]
where \( q', p' \) satisfy the same Poisson bracket relations
\[ \{q_n', q_m'\} = 0 \]
\[ \{p_n', p_m'\} = 0 \]
\[ \{q_n', p_m'\} = \delta_{mn} \]

note
\[ \frac{dq_n'}{dt} = \sum_m \left( \frac{\partial q_n'}{\partial p_m} \frac{dp_m}{dt} + \frac{\partial q_n'}{\partial q_m} \frac{dq_m}{dt} \right) \]
\[ = \sum_m \left( \frac{\partial q_n'}{\partial p_m} \frac{dp_m}{dt} + \frac{\partial q_n'}{\partial q_m} \frac{dq_m}{dt} \right) \]
\[ = \sum_m \left( \frac{\partial q_n'}{\partial p_m} \frac{dp_m}{dt} + \frac{\partial q_n'}{\partial q_m} \frac{dq_m}{dt} \right) \]
\[ = \sum_m \left( \frac{\partial q_n'}{\partial p_m} \frac{dp_m}{dt} + \frac{\partial q_n'}{\partial q_m} \frac{dq_m}{dt} \right) \]
\[ = \sum_m \left( \frac{\partial q_n'}{\partial p_m} \frac{dp_m}{dt} + \frac{\partial q_n'}{\partial q_m} \frac{dq_m}{dt} \right) \]
\[ = \sum_m \left( \frac{\partial q_n'}{\partial p_m} \frac{dp_m}{dt} + \frac{\partial q_n'}{\partial q_m} \frac{dq_m}{dt} \right) \]
\[ = \sum_m \left( \frac{\partial q_n'}{\partial p_m} \frac{dp_m}{dt} + \frac{\partial q_n'}{\partial q_m} \frac{dq_m}{dt} \right) \]
a similar calculation gives

\[ \frac{dp}{dt} = -\frac{\partial H}{\partial q} \]

This shows that canonical transformations preserve the form of Hamilton's equations.

Note

\[ dpdq = \left| \frac{\partial p}{\partial p'} \frac{\partial q}{\partial p'} \right| dp'dq' \]

\[ = \left( \frac{\partial p}{\partial q} \frac{\partial q}{\partial q'} - \frac{\partial p}{\partial q'} \frac{\partial q}{\partial q'} \right) \]

\[ = dqdp \frac{dq'}{dq} \]

\[ = dp'dq' \]

For one degree of freedom, this shows that canonical transformations preserve volumes in phase space.
there are a number of techniques for constructing canonical transformations

Time evolution is a canonical transformation

$\{ F(t), G(t) \} = \{ F(0), G(0) \}$

$e^{D_h t} \{ F(0), G(0) \} = \{ e^{D_h t} F(0), e^{D_h t} G(0) \}$

$e^{D_h t} \{ F(0), G(0) \} = \{ e^{D_h t} F(0), e^{D_h t} G(0) \}$

$\{ Q_n(t), Q_m(t) \} = e^{D_h t} \{ Q_n(0), Q_m(0) \}$

$\{ Q_n(t), P_m(t) \} = e^{D_h t} \{ Q_n(0), P_m(0) \} = \delta_{nm}$

$\{ P_n(t), P_m(t) \} = e^{D_h t} \{ P_n(0), P_m(0) \} = 0$

example: harmonic oscillity

$X(t) = \cos \omega t \, X(0) + \frac{1}{\sqrt{m k}} \sin \omega t \, P(0)$

$P(t) = \cos \omega t \, P(0) - \frac{1}{\sqrt{m k}} \sin (\omega t) \, X(0)$

$\{ X(t), P(t) \} =$

$\{ \cos \omega t \, X(0) + \frac{1}{\sqrt{m k}} \sin \omega t \, P(0) \}$

$\{ \cos \omega t \, P(0) - \frac{1}{\sqrt{m k}} \sin \omega t \, X(0) \} =$

$\cos^2 \omega t \, \{ X(0), P(0) \} - \sin^2 \omega t \, \{ P(0), X(0) \}$

$\{ X(0), P(0) \}$
from what we showed earlier
\[ dp(t) \, dx(t) = dp(0) \, dx(0) \]
which means a small area in \( x-p \) plane moves without changing volume
oscillate

we see that the group of initial conditions moves without growing or shrinking in volume,

while we showed this for 1 degree of freedom, the result holds for any number of \( q \)'s \( p \)'s

this indicates that there is something special about using coordinates and generalized moments