Lecture 4

Summary
review - damped oscillators
driven damped oscillators
resonance
conservation laws

Last time

\[ m \frac{d^2x}{dt^2} = -kx - \lambda \frac{dx}{dt} \quad (\text{Newton's 2nd law}) \]

try a solution of the form \( Ae^{pt} \)
inseting this in the equation

\[ (mp^2 + k + \lambda p) Ae^{pt} = 0 \]

for a non zero solution we must have

\[ p^2 + \frac{\lambda}{m} p + k \frac{k}{m} = 0 \]
	his is a quadratic equation for \( p \)

\[ p = \frac{1}{2} \left( -\frac{\lambda}{m} \pm \sqrt{\left(\frac{\lambda}{m}\right)^2 - 4 \frac{k}{m}} \right) \]

\[ = -\frac{\lambda}{2m} \pm \sqrt{\frac{\lambda^2}{4m^2} - \frac{k}{m}} \]

define \( \gamma = \frac{\lambda}{2m} \quad \omega_0^2 = \frac{k}{m} \)
\[ P = -\gamma \pm \sqrt{\gamma^2 - \omega^2} \]

If \( \gamma > \omega \) then

\[ P_+ = -\gamma \quad < 0 \]
\[ P_- = -\gamma(1 - \sqrt{1 - \left(\frac{\omega}{\gamma}\right)^2}) < 0 \]

\[ x(t) = Ae^{-P_+t} + Be^{-P_-t} \]

If \( \gamma < \omega \)

\[ P = -\gamma \pm i \omega \quad \omega = \sqrt{\omega^2 - \gamma^2} \]

\[ x(t) = e^{-\gamma t} (A e^{i\omega t} + B e^{-i\omega t}) \]

If \( \gamma > \omega \) both solutions decay exponentially.

If \( \gamma < \omega \), both solutions oscillate while the amplitude of the oscillation decays exponentially.

There is a special case \( \gamma = \omega \). In this case, the guess \( x(t) = Ae^{P_+t} \) gives only one solution \( x(t) = Ae^{-\gamma t} \) so it is impossible to satisfy the initial conditions with this choice.

By instead solving the pair of coupled first order equations we get a second solution of the form

\[ x(t) = te^P \]
In the critically damped case

\[
\frac{d^2x}{dt^2} = -\gamma x - \gamma \frac{dx}{dt}
\]

\[
\frac{d}{dt} (t e^{-\gamma t}) = e^{-\gamma t} - \gamma te^{-\gamma t}
\]

\[
\frac{d^2}{dt^2} (t e^{-\gamma t}) = \frac{d}{dt} (e^{-\gamma t} - \gamma te^{-\gamma t})
\]

\[
= -\gamma^2 e^{-\gamma t} + \gamma^2 t e^{-\gamma t} - \gamma^2
\]

Multiplying the first equation by \(-\gamma\) gives

\[-\gamma \frac{dx}{dt} = -\gamma e^{-\gamma t} + \gamma^2 t e^{-\gamma t}
\]

\[
\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \gamma x = -2\gamma e^{-\gamma t} + \gamma^2 t e^{-\gamma t} - \gamma^2 e^{-\gamma t} + \gamma^2 t e^{-\gamma t} = 0
\]

which shows that in the critically damped case

\[x(t) = A e^{-\gamma t} + Bte^{-\gamma t}\]

case of a driven oscillator

\[m \frac{d^2x}{dt^2} + \lambda \frac{dx}{dt} + x = F \cos(\omega t)\]
If \( \mathbf{X}(t) = \mathbf{X}_1(t) + \mathbf{X}_2(t) \) where

\[
m \frac{d^2 \mathbf{X}_1}{dt^2} + \lambda \frac{d \mathbf{X}_1}{dt} + \mathbf{X}_1 = 0
\]

\[
m \frac{d^2 \mathbf{X}_2}{dt^2} + \lambda \frac{d \mathbf{X}_2}{dt} + \mathbf{X}_2 = F \cos(\omega t)
\]

The clearly \( \mathbf{X}(t) = A \mathbf{X}_1(t) + \mathbf{X}_2(t) \) is a solution for any \( A \). We have already solved for \( \mathbf{X}_1(t) \) — there are two independent solutions. Therefore we need one specific solution to the inhomogeneous equations.

Since (for real \( F \))

\[
F \cos(\omega t) = \text{Re}(F e^{i\omega t})
\]

we can solve the complex equation

\[
m \frac{d^2 \mathbf{Z}}{dt^2} + \lambda \frac{d \mathbf{Z}}{dt} + \mathbf{Z} = F e^{i\omega t}
\]

The real part of this equation is a solution to (1). The imaginary part is a solution to

\[
m \frac{d^2 \mathbf{X}}{dt^2} + \lambda \frac{d \mathbf{X}}{dt} + \mathbf{K} \mathbf{X} = F \sin(\omega t)
\]
Try $z(t) = Ae^{i\omega t}$

$$(m(-\omega^2)A + \lambda(i\omega)A + kA)e^{i\omega t} = Fe^{i\omega t}$$

Solving for $A$

$$(-\omega^2m + i\omega\lambda + k)Ae^{i\omega t} = Fe^{i\omega t}$$

canceling $e^{i\omega t}$ we get

$$A = \frac{F}{k - \omega^2m + i\omega\lambda} = \frac{F/m}{k/m - \omega^2 + i\omega\lambda/m}$$

define $k/m = \omega_0^2$ (frequency in the absence of dissipation or driving)

$$A = \frac{F/m}{\omega_0^2 - \omega^2 + i\omega\lambda/m}$$

to separate the real and imaginary parts multiply numerator and denominator by

$$1 = \frac{\omega_0^2 - \omega^2 - i\omega\lambda/m}{\omega_0^2 - \omega^2 + i\omega\lambda/m} \quad \text{and} \quad \Lambda = \frac{F/m}{(\omega_0^2 - \omega^2)^2 + \omega^2\lambda^2/m^2} \cdot (\omega_0^2 - \omega^2 + i\omega\lambda/m)$$

$$= \frac{F/m}{(\omega_0^2 - \omega^2)^2 + 4\omega^2\lambda^2} \cdot (\omega_0^2 - \omega^2 - 2i\omega\lambda)$$
The complex solution is

\[ x(t) = \frac{F/m}{(\omega_0^2 - \omega^2)^2 + 4\omega_1^2\gamma^2} \left( \frac{\omega_0^2 - \omega_0^2 - 2i\gamma \omega}{\omega_0^2 - \omega^2} \right) e^{i\omega t} \]

\[ = \frac{F/m}{(\omega_0^2 - \omega^2)^2 + 4\omega_1^2\gamma^2} \left( \frac{\omega_0^2 - \omega_0^2 - 2i\gamma \omega}{\omega_0^2 - \omega^2} \right) \left( \cos \omega t + i \sin \omega t \right) \]

\[ = \frac{F/m}{(\omega_0^2 - \omega^2)^2 + 4\omega_1^2\gamma^2} \left( \omega_0^2 - \omega^2 \right) \cos \omega t + \]

\[ \frac{F/m}{(\omega_0^2 - \omega^2)^2 + 4\omega_1^2\gamma^2} \left( \omega_0^2 - \omega^2 \right) \sin \omega t \]

\[ = \left( \frac{F/m}{(\omega_0^2 - \omega^2)^2 + 4\omega_1^2\gamma^2} \left( \omega_0^2 - \omega^2 \right) \sin \omega t \right) \text{ Real part} \]

\[ - \left( \frac{F/m}{(\omega_0^2 - \omega^2)^2 + 4\omega_1^2\gamma^2} \left( \omega_0^2 - \omega^2 \right) \cos \omega t \right) \text{ Imaginary part} \]

The real part is the specific solution with driving term \( F \cos(\omega t) \)

\[ x(t) = \frac{F/m}{(\omega_0^2 - \omega^2)^2 + 4\omega_1^2\gamma^2} \left( \omega_0^2 - \omega^2 \right) \cos(\omega t) + \]

\[ \frac{F/m}{(\omega_0^2 - \omega^2)^2 + 4\omega_1^2\gamma^2} \left( \omega_0^2 - \omega^2 \right) \sin(\omega t) \]
The full solution is
\[ x(t) = e^{-\alpha t} \left( Ae^{-i\omega_1 t} + Be^{i\omega_1 t} \right) + x_s(t) \]
where \( \omega_1 = \sqrt{\omega_c^2 - \gamma^2} \), \( \omega_c \geq \gamma \)
\[ x(t) = \left( Ae^{-\beta_1 t} + Be^{\beta_1 t} \right) + x_s(t) \]
where \( \beta_1 = \gamma + \sqrt{\gamma^2 - \omega_c^2} \), \( \gamma > \omega \)
\[ \beta_2 = \gamma - \sqrt{\gamma^2 - \omega_c^2} \), \( \gamma > \omega \).
\[ x(t) = e^{-\gamma t} \left( A + B e^{\gamma t} \right) + x_s(t) \]
\( \gamma = \omega \).

In all 3 cases the \( A, B \) are used to satisfy the initial conditions. All 3 of the homogeneous solutions eventually vanish - what remains long term is the solution \( x_s(t) \).

This has the form
\[ A \cos \omega t + B \sin \omega t = \sqrt{A^2 + B^2} \left( \frac{A}{\sqrt{A^2 + B^2}} \cos \omega t + \frac{B}{\sqrt{A^2 + B^2}} \sin \omega t \right) \]
\[ = \sqrt{A^2 + B^2} \left( \cos \phi \cos \omega t + \sin \phi \sin \omega t \right) \]
\[ + \tan \phi = \frac{\sin \phi}{\cos \phi} = \frac{B}{A} \]
\[ = \sqrt{A^2 + B^2} \cos (\omega t - \phi) \]
\[
\sqrt{A^2 + B^2} = \frac{F/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2B^2}}
\]

\[
\tan \phi = \frac{2\omega}{\omega_0^2 - \omega^2}
\]

\[
X_s(t) = \frac{F/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2B^2}} \cos(\omega t - \phi)
\]

We can imagine varying \( \omega \) on \( \omega \)

* For fixed \( \omega \) the amplitude will be maximal when \( \omega_0 = \omega \)

* For fixed \( \omega \) the amplitude will be maximal when the denominator is minimal.

\[
\frac{d}{d\omega^2} \left( (\omega_0^2 - \omega^2)^2 + 4\omega^2B^2 \right) = 0
\]

\[
2(\omega_0^2 - \omega^2) + 4\omega^2 = 0
\]

\[
\omega^2 = \omega_0^2 - 2B^2
\]

(The second derivative is positive so this is a minimum.)
for small $\gamma$ the peak is sharp
for fixed $\omega^2$ the peak is at $\omega^2$
for fixed $\omega^0$ the peak is at $\omega = \omega^2 - \gamma^2$
which is close to $\omega^2$

A peak the amplitude is

for fixed $\omega^2$

$$A = \frac{F/m}{\sqrt{4\gamma^2\omega^2}} = \frac{F/m}{m} \frac{1}{2\pi \omega}$$

for fixed $\omega^0$

$$A = \frac{F/m}{\sqrt{4\gamma^2 + 4\gamma^2 (\omega^2 - \gamma^2)}} = \frac{F/m}{\sqrt{4\gamma^2 (\omega^2 - \gamma^2)}}$$

As $\gamma \to \infty$ the amplitude gets very
large.

*Interpretation of $\gamma$*

when $(\omega^2 - \omega^2)^2 = 4\gamma^2 \omega^2$ the amplitude
is down by a factor of 2 –
then

$$|\omega^2 - \omega^2| = 2\gamma \omega = (\omega + \omega_0)(\omega - \omega_0)$$
for \( w \) close to \( \omega_0 \)

\[
\frac{1}{\sqrt{(w^2 - \omega_0^2)^2 + 4\lambda^2 \omega_0^2}} = \frac{1}{\sqrt{(w - \omega_0)(w + \omega_0)^2 + 4\lambda^2 \omega_0^2}} \\
\approx \frac{1}{\sqrt{\lambda^2 (2\omega_0^2 + 4\lambda^2 \omega_0^2)}}
\]

The amplitude becomes

\[
\frac{F/m}{2\omega_0 \sqrt{2}}
\]

Thus \( \gamma \) measures half the width where the amplitude is down by a factor of \( \frac{1}{\sqrt{2}} \)

![Graph](image)

The last interesting parameter is the quality factor. It is the ratio of the peak at resonance to the peak width, \( \omega_0 \)

\[
Q = \frac{\frac{F}{m}}{\frac{1}{2\lambda \omega_0}} = \frac{\omega_0}{2\lambda}
\]
This is a measure of how sharp the peak rises.

If we have more than 1 periodic force

\[ m \frac{d^2z}{dt^2} + kZ + k \frac{dz}{dt} = F_1 e^{i\omega t} + F_2 e^{-i\omega t} \]

let \[ A_1 e^{i\omega t}, \ A_2 e^{i\omega t}, \ A_3 e^{i\omega t} \]

be solutions to each function separately. Then

\[ \left( m \frac{d^2}{dt^2} + x \frac{d}{dz} + k \right) \left( A_1 e^{i\omega t} + A_2 e^{i\omega t} + A_3 e^{i\omega t} \right) \]

\[ = F_1 e^{i\omega t} + F_2 e^{i\omega t} + F_3 e^{i\omega t} \]

This shows that the sum of the specific solutions is a specific solution to the equation with the sum of those forces.

We can still add solutions to the homogeneous equation which are needed to fix the initial conditions.
we can write this as

\[ \frac{1}{T} \int_0^T \sin \omega t - i \omega t \, dt = S_{nm} = \left\{ \begin{array}{ll} \frac{1}{n} & n = m \\ 0 & n \neq m \end{array} \right. \]

The symbol \( S_{nm} \) is called a Kronecker \( \delta \).

The series

\[ f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i \omega t} \quad (9) \]

is called a Fourier series. Note

\[ \frac{1}{T} \int_0^T f(t) dt = \sum_{n=-\infty}^{\infty} c_n \frac{1}{T} \int_0^T e^{i (n-m) \omega t} dt \]

\[ = \sum_{n=-\infty}^{\infty} c_n S_{nm} = C_m \]

\[ \therefore C_m = \frac{1}{T} \int_0^T \sin \omega t \, dt \]

We assumed that \( f(t) \) had the
form of the sum in (9). But
it turns out and continuous
function with period \( T \)
can be expressed in such a series
with \( c_m \) defined above.
A general function with period $T = \frac{2\pi}{\omega}$ does not have to be of the form $\sin \omega t$ or $\cos \omega t$.

The functions

$$\sin (n\omega t), \cos (n\omega t)$$

or

$$e^{in\omega t}, e^{-in\omega t}$$

for $n$ integers all have period $T$.

So a function of the form

$$f(t) = f_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin (n\omega t)$$

is valid and all have period $T = \frac{2\pi}{\omega}$. Note that

$$\int_{0}^{T} e^{i\omega t} e^{-im\omega t} dt = 0$$

$$\int_{0}^{\infty} e^{i\omega(n-m)t} dt = \frac{1}{i\omega(n-m)} \left( e^{i\omega(n-m)T} - 1 \right)$$

for $n=m$ and $n \neq m$. The result is:

$$\begin{cases} T & n=m \\ \frac{1}{i\omega(n-m)} \left( e^{i\omega(n-m)T} - 1 \right) & n \neq m \end{cases}$$
Thus for a general periodic function
\[ m \ddot{x} + 2\dot{x} + kx = F(t) \]
has solutions of the form
\[ x(t) = A x_1(t) + B x_2(t) + x_3(t) \]
where \( F(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega t} \)
\[ x_3(t) = \sum_{n} x_{3n}(t) \]
\[ x_{3n}(t) = \frac{c_n}{k/m - (n\omega)^2 + i\omega \gamma/m} e^{i\omega t} \]

**Impulse Forces** - It is possible to exert force on the particle over a very short time. In this case

1. The position of the particle at the time the force is applied does not change.
2. \( \int F dt = \) work done on the particle by the force

\[ \begin{align*}
\int F dt &= \int m \frac{d^2 x}{dt^2} \, dt + \int \frac{d}{dt} (mv) \\
&= \int m (V(t+\epsilon) - V(t)) \\
&= \frac{1}{m} \int F dt = \text{change in velocity}
\end{align*} \]
the quantity $I = \int F \, dt$ is called an impulse - the change in velocity is $I/m$

for a damped oscillator at rest at the origin (stable equilibrium point)

we have a specific solution

where the particle is initially at rest

$$X(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{w} \frac{I}{m} e^{-\frac{t}{w}} \sin \omega t & t \geq 0 \end{cases}$$

Note

$$X(0) = \frac{1}{w} \frac{I}{m} (-r, 0 + \omega, 1) = \frac{I}{m}$$

To this we can add solutions to the homogeneous equation

$$X(t) = AX_1(t) + BX_2(t) + X_I(t)$$

which satisfies the equation

$$\dot{X} + \frac{h}{m} X + \frac{1}{m} \dot{X} = F(t)$$

where $F(t)$ is the impulse force
A and B can be used to set the initial position and velocity - in this case the position does not change at time \( t = 0 \) but the velocity jumps by \( Y_m \) at \( 0 \).

\[
G(t-t_0) = \begin{cases} 
0 & t < t_0 \\
\frac{1}{wm}e^{-(t-t_0)/\tau}\sin\omega(t-t_0) & t \geq t_0
\end{cases}
\]

\[
\int G(t-t')F(t')dt' + AX_1(t) + BX_2(t) \sim \text{Impulse}
\]

In this way we can express the effects of a general force as a sequence of impulses.

The function \( G(t-t_0) \) is called a Green's function for this damped oscillatory
Two body collisions in 1 dimension

Since $F_{12} = -F_{21}$ we have by the third law

$$m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = 0$$

$$\frac{d}{dt} (m_1 \dot{x}_1 + m_2 \dot{x}_2) = 0$$

This says that the total linear momentum of the system is conserved.

In a collision the particle interact and then move freely after collision

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

If the force during the collision is conservative then energy is conserved.

If the particles are moving freely then the energy is kinetic:

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$$

$$\frac{1}{2} m_1 (v_1^2 - v_1'^2) + \frac{1}{2} m_2 (v_2^2 - v_2'^2) = 0$$

$$m_1 (v_1 - v_1') + m_2 (v_2 - v_2') = 0$$
\[
\frac{1}{2} m_1 (v_i - v'_i)(v_i + v'_i) = -\frac{1}{2} m_2 (v_2 - v'_2)(v_2 + v'_2)
\]

\[
m_1 (v_i - v'_i) = -m_2 (v_2 - v'_2)
\]

given

\[
\frac{1}{2} (v_i + v'_i) = \frac{1}{2} (v_2 + v'_2)
\]

\[
\frac{1}{2} m_1 v_i + m_2 v_2 = m_1 v'_i + m_2 v'_2
\]

these are 2 equations in 2 unknowns so we can solve for \(v_i, v_2\) in terms of \(v'_i, v'_2\)

\[
m_1 v_i - m_1 v'_i = m_1 v'_2 - m_1 v'_1
\]

\[
m_1 v_i + m_1 v'_i = m_2 v'_2 + m_1 v'_1
\]

adding

\[
2m_1 v_i + (m_2 - m_1) v_2 = (m_1 + m_2) v'_2
\]

\[
v_2' = \frac{m_2 - m_1}{m_2 + m_1} v_2 + \frac{2m_1}{m_1 + m_2} v_i
\]

\[
v_i' = \frac{m_1 - m_2}{m_1 + m_2} v_i + \frac{2m_2}{m_1 + m_2} v_2
\]

we can also have inelastic collisions where kinetic energy is lost in the collision

in the elastic case the magnitude of the relative momentum is the same.
In an inelastic collision the relative velocities must be reduced. We write this as

\[ (v_1 - v_2) = e (v_2' - v_1') \]

\( e = 1 \) for elastic collisions - \( e = 1 \)

\( \text{for inelastic collisions} \quad 0 \leq e < 1 \)

we still have

\[ m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \]

we have 2 equations in 2 unknowns that can be solved for \( v_1', v_2' \) in terms of \( v_1, v_2 \)

\[ \frac{m_1}{e} (v_1 - v_2) = m_1 v_1' - m_1 v_1 \]

\[ m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \]

adding

\[ m_1 (1 + \frac{1}{e}) v_1 + m_2 (1 - \frac{1}{e}) v_2 = (m_1 + m_2) v_2' \]

\[ v_2' = \frac{m_1}{m_1 + m_2} \left( 1 + \frac{1}{e} \right) v_1 + \frac{m_2}{m_1 + m_2} \left( 1 - \frac{1}{e} \right) v_2 \]

\[ v_1' = \frac{m_1}{m_1 + m_2} \left( 1 + \frac{1}{e} \right) v_2' + \frac{m_1}{m_1 + m_2} \left( 1 - \frac{1}{e} \right) v_1 \]