Lecture 7

Potentials + conservative forces

Example

Angular momentum

other coordinate systems

calculus of variations

---

Summary - In 3 dimensions

\[
\frac{d}{dt} (T+V) = 0
\]

\[T+V = E \text{ conserved}\]

\[
\vec{F} = -\nabla V = -(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z})V(x,y,z)
\]

This will be true if

\[
\nabla \times \vec{F} = 0 \quad (\text{this is called curl of } \vec{F})
\]

\[
V(r) = -\int_{r_0}^{r} \vec{F}(\vec{r}(s)) \cdot \frac{d\vec{r}}{ds} \, ds \quad \vec{r}(1) = \vec{r} \quad \vec{r}(0) = \vec{r}_0
\]

The work done in moving the particle from \(\vec{r}_0\) to \(\vec{r}\) against the force \(\vec{F}\) is independent of the path between \(\vec{r}_0\) and \(\vec{r}\).

3 boxed equations are equivalent
example: particle in a constant gravitational field

* choose coordinate system
  choose origin to be at the initial point of the particle
  \( \vec{r}(0) = (0, 0, 0) = \hat{i}0 + \hat{j}0 + \hat{k}0 \)
  choose the z axis up.
  choose the x axis so the initial velocity is in the positive x part of the x-z plane
  \( \vec{r}(0) = (V_x 0 V_z) = \hat{i}V_x + \hat{j}0 + \hat{k}V_z \)
  with \( V_x = 0 \)

  \( \vec{F} = (0, 0, -mg) = -mg \hat{k} \)

  \( V = mgz \)

  \( -\vec{V}V = (\hat{i}\frac{\partial}{\partial x} - \hat{j}\frac{\partial}{\partial y} - \hat{k}\frac{\partial}{\partial z}) (mgz) \)

  \( = \hat{i}0 + \hat{j}0 - mg \hat{k} = \vec{F} \)

\[
\begin{align*}
  V_x &= V \cos \theta \\
  V_z &= V \sin \theta \\
  V &= \sqrt{V_x^2 + V_z^2}
\end{align*}
\]
Newton's second law

\[ m \frac{d^2 \vec{r}}{dt^2} = \vec{F} \]

\[ m \ddot{x} = 0 \]
\[ m \ddot{y} = 0 \]
\[ m \ddot{z} = -mg \]

These are easily integrated

\[ X(t) = X(0) + V_x(0) t \]
\[ Y(t) = Y(0) + V_y(0) t \]
\[ Z(t) = Z(0) + V_z(0) t - \frac{1}{2} gt^2 \]

Using the initial conditions:

\[ X(t) = V \cos \theta \ t \]
\[ Y(t) = 0 \]
\[ Z(t) = V \sin \theta \ t - \frac{1}{2} gt^2 \]

Using

\[ t = \frac{x}{V \cos \theta} \]

can be used to get the trajectory,

\[ Z(x) = X \tan \theta - \frac{c^2}{2V^2 \cos^2 \theta} x^2 \]
we can also find the range

\[ z(t) = 0 = t \left( v \sin \theta - \frac{1}{2} g t \right) \]

\[ t = 0 \quad t = \frac{2v \sin \theta}{g} \]

\[ x(t) = v \cos \theta \left( \frac{2v \sin \theta}{g} \right) = \frac{v^2}{g} \sin \theta \cos \theta \]

\[ = \frac{v^2}{g} \sin 2\theta \]

the maximum range is \( \frac{v^2}{g} \) when

\[ 2\theta = \frac{\pi}{2} \quad \theta = \frac{\pi}{4} \]

In reality there is some dissipation. The simplest version is

\[ F = -mg \hat{k} - \gamma \dot{r} \]

which gives a force that tries to slow down the particle. The equations of motion become

\[ m \frac{d^2x}{dt^2} = -\gamma \frac{dx}{dt} \]

\[ m \frac{d^3y}{dt^3} = -\gamma \frac{dy}{dt} \]

\[ m \frac{d^2z}{dt^2} = -\gamma \frac{dz}{dt} - mg \]

Let \( \chi = \frac{y}{m} \)
\[
\frac{d^2 x}{dt^2} + \lambda \frac{dx}{dt} = 0 \\
\frac{d^2 y}{dt^2} + \lambda \frac{dy}{dt} = 0 \\
\frac{d^2 z}{dt^2} + \lambda \frac{dz}{dt} + q = 0
\]

To solve this we use an integrating factor (multiply by \(e^{\lambda t}\)) to get

\[
\frac{d}{dt} \left( e^{\lambda t} \frac{dx}{dt} \right) = 0 \\
\frac{d}{dt} \left( e^{\lambda t} \frac{dy}{dt} \right) = 0 \\
\frac{d}{dt} \left( e^{\lambda t} \frac{dz}{dt} + \frac{q}{\lambda} e^{\lambda t} \right) = 0
\]

This gives 3 constants

\[
e^{\lambda t} \frac{dx}{dt} = V_x(t) = v \cos \omega
\]

\[
e^{\lambda t} \frac{dy}{dt} = V_y(t) = 0
\]

\[
e^{\lambda t} \frac{dz}{dt} + \frac{q}{\lambda} e^{\lambda t} = V \sin \omega + \frac{q}{\lambda}
\]

This gives

\[
\frac{dx}{dt} = V \cos \omega e^{-\lambda t}
\]

\[
\frac{dy}{dt} = 0
\]
\[ \frac{dz}{dt} = -\frac{a}{\lambda} + \left( v \sin \theta + \frac{a}{\lambda} \right) e^{-\lambda t} \]

Integrating:

\[ x(t) = x(0) - \frac{v}{\lambda} \cos \theta (e^{-\lambda t} - 1) \]
\[ y(t) = -\frac{v}{\lambda} \cos \theta (e^{-\lambda t} - 1) \]
\[ z(t) = z(0) - \frac{a}{\lambda} t + \frac{1}{\lambda} (v \sin \theta + \frac{a}{\lambda}) (e^{-\lambda t} - 1) \]
\[ = -\frac{a}{\lambda} t - \frac{1}{\lambda} (v \sin \theta + \frac{a}{\lambda}) (e^{-\lambda t} - 1) \]

It is also possible to eliminate time:

\[ e^{-\lambda t} - 1 = -\frac{\lambda x}{v} \frac{1}{\cos \theta} \]
\[ -\lambda t = \ln \left( 1 - \frac{\lambda x}{v} \frac{1}{\cos \theta} \right) \]
\[ t = -\frac{1}{\lambda} \ln \left( 1 - \frac{\lambda x}{v} \frac{1}{\cos \theta} \right) \]

Torque and angular momentum

given an origin:

\[ T = (\vec{r} \times \vec{F}) = \hat{i} (x F_z - z F_y) + \hat{j} (z F_y - x F_z) + \hat{k} (x F_y - y F_x) \]
The angular momentum

\[ \mathbf{\tau} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m \mathbf{\dot{r}} \]

\[ \mathbf{\tau} = m \mathbf{\hat{r}} \left( y \mathbf{\dot{z}} - z \mathbf{\dot{y}} \right) + \\
  m \mathbf{\dot{\hat{r}}} \left( z \mathbf{\dot{x}} - x \mathbf{\dot{z}} \right) + \\
  m \mathbf{\hat{r}} \left( x \mathbf{\dot{y}} - y \mathbf{\dot{x}} \right) \]

Note that

\[ \frac{d\mathbf{\tau}}{dt} = \frac{d}{dt} \left( \mathbf{r} \times m \mathbf{\dot{r}} \right) = \\
  = m \left( \mathbf{\dot{r}} \times m \mathbf{\ddot{r}} \right) + \mathbf{r} \times \left( m \mathbf{\ddot{r}} \right) \\
  = \mathbf{r} \times m \mathbf{\ddot{r}} \\
  = \mathbf{\tau} \]

If there is no torque about the origin - then the angular momentum about that point is conserved.
If $\vec{J}$ is conserved then both $\vec{r}$ and $\vec{v}$ remain in the plane $\perp$ to $\vec{J}$

\[ \vec{V} = \vec{V}_{\parallel} + \vec{V}_{\perp} \]

\[ \frac{1}{2} \vec{r} \vec{V}_{\perp} \Delta t = \text{Area swept out in time } \Delta t \]

\[ \frac{dA}{dt} = \frac{1}{2m} |\vec{J}| \]

This is called Kepler's 2nd law - it is simply a restatement of angular momentum conservation

A force is called central if

\[ \vec{F}(\vec{r}) = \vec{r} \vec{g}(\vec{r}) \]

In this case

\[ \vec{F} = \vec{r} \times \vec{F}(\vec{r}) = \vec{r} \times \vec{r} \vec{g}(\vec{r}) = 0 \]
The means

1. angular momentum is conserved in central forces;
2. motion under the influence of central forces are in a plane.

In some applications it is useful to use coordinates that are not cartesian. We already did this in a simple pendulum. A case of interest is spherical coordinates that are useful for problems with spherical symmetry

\[ r = \sqrt{x^2 + y^2 + z^2} \]
\[ \cos \theta = \frac{z}{r} \]
\[ \tan \phi = \frac{y}{x} \]
\[ x = r \sin \theta \cos \phi \]
\[ y = r \sin \theta \sin \phi \]
\[ z = r \cos \phi \]
\[ x = \dot{r} \sin \omega \cos \phi \\
- r \cos \omega \cos \phi \dot{\theta} \\
- r \sin \omega \sin \phi \dot{\phi} \\
\]
\[ y = \dot{\rho} \sin \phi \\
- r \cos \omega \sin \phi \dot{\theta} \\
r \sin \omega \cos \phi \dot{\phi} \\
\]
\[ z = \dot{r} \cos \omega - r \sin \omega \dot{\phi} \]

Adding (the mixed terms cancel)
\[ T = \frac{1}{2} (m \dot{x}^2 + m \dot{y}^2 + m \dot{z}^2) \]
\[ = \frac{1}{2} m \left( \dot{r}^2 \left( \sin^2 \omega \cos^2 \phi + \sin^2 \omega \sin^2 \phi + \cos^2 \omega \right) \right) \]
\[ + \left( \dot{\rho}^2 \right) \left( \cos^2 \omega \left( \cos^2 \phi + \sin^2 \phi \right) + \sin^2 \omega \right) \]
\[ + \dot{r}^2 \dot{\phi}^2 \left( \sin^2 \omega \sin^2 \phi + \sin^2 \omega \cos^2 \phi \right) \]
\[ = \frac{1}{2} m \left( \dot{r}^2 + \dot{\rho}^2 + r^2 \sin^2 \omega \phi^2 \right) \]

We can do the same with cylindrical coordinates.
\[ x = \rho \cos \phi \]
\[ y = \rho \sin \phi \]
\[ z = z \]
\[ \rho = \sqrt{x^2 + y^2} \]
\[ \tan \phi = \frac{y}{x} \]
\[ x = \rho \cos \theta - \rho \sin \theta \dot{\phi} \]
\[ y = \rho \sin \theta + \rho \cos \theta \dot{\phi} \]
\[ z = z \]
\[ T = \frac{1}{2} m (x'^2 + y'^2 + z'^2) \]
\[ = \frac{1}{2} m (\rho'^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2) \]

Calculus of Variations

What is the shortest path between 2 points in a plane?

1. Pick a curve \( y(x) \)

\( y(x_1) = y_1 \quad \Delta S = \Delta x^2 + \Delta y^2 \quad \frac{dy}{dx} = \frac{\Delta y}{\Delta x} \)

\( y(x_2) = y_2 \)

\[ L = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{dx^2 + dy^2} \]
\[ = \int_{x_1}^{x_2} \sqrt{dx^2 + (\frac{dy}{dx})^2} \, dx \]
\[ = \int_{x_1}^{x_2} \sqrt{1 + (\frac{dy}{dx})^2} \, dx \]

We don't know \( y(x) \).
Let \( Y_0(x) \) be the curve that gives the shortest distance between \((x_1, y_1)\) and \((x_2, y_2)\).

Consider

\[
SY(x) = Y(x) - Y_0(x) \quad \text{for an arbitrary curve}
\]

Let \( x = x_1 \leq x_2 \)

\[
SY(x_1) = Y(x_1) - Y_0(x_1) = y_1 - y_1 = 0
\]
\[
SY(x_2) = Y(x_2) - Y_0(x_2) = y_2 - y_2 = 0
\]

Consider

\[
Y(x, s) = Y_0(x) + s \cdot SY(x)
\]
\[
Y(x, s) = y_1 \quad Y(x, s) = y_2
\]

and define

\[
L(s, SY) = \int_{x_1}^{x_2} \sqrt{1 + \left( \frac{dy(x, s)}{dx} \right)^2} \, dx
\]

If every \( SY \) makes \( L \) increase, then

\[
0 = \frac{dL}{ds}(s, SY) = \frac{d}{ds} \int_{x_1}^{x_2} \sqrt{1 + \left( \frac{dy}{ds} \right)^2} \, dx \bigg|_{s = 0}
\]
\[
\frac{d}{ds} \int_{x_1}^{x_L} \sqrt{1 + \left( \frac{dy_0}{dx} + s \cdot \frac{dY}{dx} \right)^2} \, dx = 0
\]

setting \( s = - \)

\[
\int_{x_1}^{x_L} \frac{dY}{dx} \frac{d}{dx} (SY) \, dx
\]

integrate by parts, \( \frac{d}{dx} (SY) \)

\[
\int_{x_1}^{x_L} \frac{d}{dx} \left( \frac{dy_0}{dx} \frac{d}{dx} \frac{dY}{dx} \right) \, dY + \frac{dY}{dx} \bigg|_{x_1}^{x_L} \, SY \bigg|_{x_1}^{x_L}
\]

must vanish for all \( SY(x) \)

\[
\frac{d}{dx} \left( \frac{dy_0}{dx} \frac{d}{dx} \frac{dY}{dx} \right) = 0
\]

\[
(dy_0 \, dy) = c^2 \left( 1 + \left( \frac{dy}{dy} \right)^2 \right)
\]

\[
\left( \frac{dy}{dy} \right)^2 \left( 1 - c^2 \right) = c^2
\]
\[
\left(\frac{dy}{dx}\right)^2 = \frac{c^2}{1-c^2} = \text{constant}
\]

\[
\left(\frac{dy}{dx}\right) = \text{constant} = \frac{y_2 - y_1}{x_2 - x_1}
\]

\[y(x) = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)
\]

which is the equation for a straight line.

Consider \( f(xy) \)

\[
\frac{d}{ds} f(x_0 + s \bar{a}) = \left( \frac{\partial f}{\partial x}(x_0) a_x + \frac{\partial f}{\partial y}(x_0) a_y \right) = 0
\]

For this to vanish for all \( \bar{a} \), means

\[
\frac{\partial f}{\partial x}(x_0) = \frac{\partial f}{\partial y}(x_0) = 0
\]

calculus of variations is an infinite dimensional version of this.