Lecture 9

Variational Calculus

Last time:

what curve is the shortest distance between 2 points

1 construct a distance functional

\[ L[y_1, x_0, x_1] = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \]

Let \( y_0(x) \) = curve of minimal length
\( y_1(x) \) = any other curve with same endpoints

\( \delta y(x) = y_1(x) - y_0(x) \)

\[ \frac{d}{ds} L[y_0 + s \delta y, x_0, x_1] \bigg|_{s=0} = 0 \]

gives differential equation for \( y(x) \)

2 consider \( s(\bar{r}) \). If we want to find a maximum or minimum consider

\[ \frac{d}{ds} s(\bar{r} + s\bar{a}) \bigg|_{s=0} = \nabla s(\bar{r}_0) \cdot \bar{a} \]
for this to be a maximum the
function of $s$ should be a maximum
at $s=0$ for any $\bar{a}$

For functionals $\bar{a}$ is replaced by $SY(x)$

In the most interesting cases
we seek a curve $q_1(x), q_N(x)$
that minimizes or maximizes
a functional that depends on
$q_1(x), q_N(x), \frac{dq_1}{dx}(x), \ldots, \frac{dq_N}{dx}(x)$

Let $\ddot{q}(x) = q_1(x), \ldots, q_N(x)$
$\dot{q}(x) = \frac{dq_1}{dx}(x), \ldots, \frac{dq_N}{dx}(x)$

consider a functional of the
form

$$F[q, \ddot{q}_f, \ddot{q}_i] = \int_{\lambda_1}^{\lambda_2} \mathcal{L}(\dot{q}(x), \ddot{q}(x), x) \, dx$$

where

$\ddot{q}(x_1) = \ddot{q}_f, \ddot{q}(x_2) = \ddot{q}_i$
we would like to find a curve $\bar{q}(u)$ subject to $\bar{q}(a) = \bar{q}_f$, $\bar{q}(a) = \bar{q}_r$ that makes this functional extremal.

To solve this problem

1. Let $\bar{q}_0(u)$ be the path that makes this functional extremal.
2. Let $\bar{q}, u$ be any other path with the same endpoints.
3. Let $s\bar{q}(u) = \bar{q}(u) - \bar{q}_0(u)$.

Consider

$$F [ \bar{q}_0 + s\bar{q}, \bar{q}_f, \bar{q}_r ] =$$

$$\int_{a_1}^{a_2} \bar{q} \left( \left( \bar{q}_0(u) + s\bar{q}(u), \bar{q}_f(u) + s\bar{q}^\prime(u), \bar{q}_r(u) \right) \right) \, du =$$

$\bar{q}_0(u)$ will be a stationary point of this functional if

$$\frac{d}{ds} F [ \bar{q}_0 + s\bar{q}, \bar{q}_f, \bar{q}_r ] \bigg|_{s=0} = 0$$

for all $\bar{q}$ satisfying $s\bar{q}(a_1) = s\bar{q}(a_2) = 0$.
(Note setting \( s = 0 \) means that we are evaluating this functional at \( \bar{\eta}_0 (u) \))

\[
\frac{d}{ds} \mathcal{F} [ \bar{\eta}_0 + s \bar{q}_1, q_1, q_F ] = \\
\int_{\lambda_1}^{\lambda_2} \left( \sum_{i=1}^{\bar{n}_i} \frac{\partial \mathcal{F}}{\partial q_i} (\bar{\eta}_0, \bar{q}_i, \bar{q}_1, u) \right) \frac{d}{du} s q_i (u) + \sum_{i=1}^{\bar{n}_i} \frac{\partial \mathcal{F}}{\partial \dot{q}_i} (\bar{\eta}_0, \bar{q}_i, \bar{q}_1, u) \frac{d}{du} s \dot{q}_i (u) \right) \, du
\]

Integrating the second term by parts gives

\[
\int_{\lambda_1}^{\lambda_2} \left( \sum_{i=1}^{\bar{n}_i} \left( \frac{\partial \mathcal{F}}{\partial q_i} (\bar{\eta}_0, \bar{q}_i, \bar{q}_1, u) \right) - \frac{d}{du} \left( \frac{\partial \mathcal{F}}{\partial \dot{q}_i} (\bar{\eta}_0, \bar{q}_i, \bar{q}_1, u) \right) \right) \delta q_i (u) \\
+ \sum_{i=1}^{\bar{n}_i} \left[ \frac{\partial \mathcal{F}}{\partial \dot{q}_i} (\bar{\eta}_0, \bar{q}_i, \bar{q}_1, u) \delta q_i (u) \right]_{\lambda_1}^{\lambda_2}
\]

The last term vanishes because \( \delta q (\lambda_1) = \delta q (\lambda_2) = 0 \).

Since \( \delta q_i (\lambda) \) is arbitrary - it follows that the coefficient of each \( \delta q_i (u) \) vanishes.

\[
0 = \frac{d}{d\lambda} \left( \frac{\partial \mathcal{F}}{\partial q_i} (\bar{\eta}_0, \bar{q}_i, \bar{q}_1, u) \right) - \frac{\partial \mathcal{F}}{\partial \dot{q}_i} (\bar{\eta}_0, \bar{q}_i, \bar{q}_1, u)
\]

\( i = 1, \ldots, n \).

These are called the Euler-Lagrange equations.
This gives a system of $N$ second order differential equations for the curve $\vec{q}(\lambda)$ that make the functional extremal.

The equations are called the Euler-Lagrange equations.

In connection with classical mechanics - Hamilton's principle.

A particle traveling from $\vec{r}(t_1) = \vec{r}_1$ to $\vec{r}(t_F) = \vec{r}_F$ in time $t_F - t_1$ will follow a path that makes the action functional stationary

$$A[r_1, \dot{r}_1, r_F, \dot{r}_F] = \int_{t_1}^{t_F} L(\vec{r}(t), \dot{\vec{r}}(t), t) \, dt$$

where $L = T - V$ is called the Lagrangian of the system.
For
\[
T = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m (x'^2 + y'^2 + z'^2)
\]
\[V = V(x, y, z)\]
\[L = \frac{1}{2} m (x'^2 + y'^2 + z'^2) - V(x, y, z)\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = m \ddot{x} + \frac{\partial V}{\partial x} = 0
\]
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} = m \ddot{y} + \frac{\partial V}{\partial y} = 0
\]
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{z}} \right) - \frac{\partial L}{\partial z} = m \ddot{z} + \frac{\partial V}{\partial z} = 0
\]

we can write these equations in vector form

\[
m \frac{d^2 \vec{r}}{dt^2} = -\nabla V
\]

which is Newton's second law.

The Euler-Lagrange equations are called Lagrange's equations in this case.

One of the values of this formalism is that it can be used with non-cartesian coordinates.
example

simple pendulum

\[ \frac{1}{2} m \dot{r}^2 = \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) \]

for a pendulum with fixed \( r = R \), \( \dot{r} = 0 \) and the kinetic energy is

\[ T = \frac{1}{2} m R^2 \left( \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right) \]

\[ V = m q y = m q (-R \cos \theta) = -m q R \cos \theta \]

\[ L = T - V = \frac{1}{2} m R^2 \left( \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right) + m q R \cos \theta \]

we can immediately write down the Lagrange's equations

\[ \frac{\partial L}{\partial \dot{\theta}} = m R^2 \ddot{\theta} \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) = m R^2 \ddot{\theta} \]

\[ \frac{\partial L}{\partial \dot{\phi}} = m R^2 \sin \theta \cos \theta \dot{\phi}^2 - m q R \sin \theta \]

\[ \frac{\partial L}{\partial \phi} = m R^2 \sin^2 \theta \dot{\phi} \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = 2 R^2 \sin \theta \cos \theta \dot{\phi}^2 + m R^2 \sin^2 \theta \ddot{\phi} \]
The equations become

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial \phi} = \text{const} \]

\[ m R^2 \sin^2 \theta \dot{\phi} = \text{const} + \]

\[ m R^2 \ddot{\phi} + m g R \sin \theta - m R^2 \sin \theta \cos \theta \dot{\phi}^2 = 0 \]

For \( \dot{\phi} = 0 \) this gives the equation that we derived by Newton's second law.

For a general conservative system in 3-dimensional polar coordinates,

\[ L = \frac{1}{2} m \left( \dot{r}^2 + \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) - V(r, \theta, \phi) \]

\[ \frac{\partial L}{\partial \dot{r}} = m \dot{r} \quad \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) = m \ddot{r} \]

\[ \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \quad \quad \frac{d}{dt} \left( m r^2 \dot{\theta} \right) = m r^2 \ddot{\theta} + 2 m r \dot{r} \dot{\theta} \]

\[ \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2 \theta \dot{\phi} \quad \quad \frac{d}{dt} \left( m r^2 \sin^2 \theta \dot{\phi} \right) = 2 m r \dot{r} \sin^2 \theta \dot{\phi} + 2 m r^2 \sin \theta \cos \theta \ddot{\phi} + m r^2 \sin^2 \theta \ddot{\phi} \]

\[ \frac{\partial L}{\partial r} = m r \dot{\theta}^2 + m r \sin \theta \dot{\phi}^2 - \frac{\partial V}{\partial r} \]

\[ \frac{\partial L}{\partial \theta} = m r^2 \sin \theta \cos \theta \dot{\phi}^2 - \frac{\partial V}{\partial \theta} \]
\[ \frac{\partial L}{\partial \dot{\phi}} = -\frac{\partial V}{\partial \phi} \]

This gives the following equations of motion:

\[ m\ddot{r} - m\ddot{\phi}^2 - m r \sin \phi \dot{\phi}^2 + \frac{\partial V}{\partial r} = 0 \]

\[ \dot{\theta} = \frac{m r^3 \ddot{\theta} + 2 m r \dot{r} \dot{\theta} - m r^2 \sin \phi \cos \phi \dot{\phi}^2 + \frac{\partial V}{\partial \theta}}{2 m r^2 \sin \phi \cos \phi \dot{\phi}} - \frac{\partial V}{\partial \phi} = 0 \]

Central force problems - Chapter 9

Example - 3 dimensional isotropic harmonic oscillator

\[ V = \frac{1}{2} k \vec{r}^2 \quad \vec{F} = -\nabla V = -k \vec{r} \]

Since the force is in the opposite direction to \( \vec{F} \) the force is central.
Newton's second law in Cartesian coordinates is

\[ m \ddot{x} = -kx \]
\[ m \ddot{y} = -ky \]
\[ m \ddot{z} = -kz \]

This behaves like 3 independent harmonic oscillators - all with angular frequency \( \omega = \sqrt{\frac{k}{m}} \).

We can write the solution in the form

\[ x(t) = c_x \sin \omega t + d_x \cos \omega t \]
\[ y(t) = c_y \sin \omega t + d_y \cos \omega t \]
\[ z(t) = c_z \sin \omega t + d_z \cos \omega t \]

We can combine these three equations into one vector equation

\[ \vec{r}(t) = \vec{c} \sin \omega t + \vec{d} \cos \omega t \]

where

\[ \vec{c} = c_x \hat{i} + c_y \hat{j} + c_z \hat{k} \]
\[ \vec{d} = d_x \hat{i} + d_y \hat{j} + d_z \hat{k} \]
In this case

\[ \ddot{p} = m \frac{d\ddot{r}}{dt} = m \ddot{c} \omega \cos \omega t - m \ddot{d} \omega \sin \omega t \]

\[ \ddot{f} = \ddot{r} \times \dot{p} = \]

\[ = (\ddot{c} \sin \omega t + \ddot{d} \cos \omega t) \times (m \ddot{c} \omega \cos \omega t - m \ddot{d} \omega \sin \omega t) \]

\[ = -m \omega \dddot{c} \dddot{d} \sin^2 \omega t + m \omega \dddot{d} \dddot{c} \cos^2 \omega t \]

using \[ \dddot{c} \dddot{d} = -\dddot{d} \dddot{c} \] gives

\[ = -\dddot{d} \dddot{c} \omega (\cos^2 \omega t + \sin^2 \omega t) \]

\[ = -\dddot{d} \dddot{c} \omega \]

we see immediately that this is independent of time and hence vector is perpendicular to both \[ \ddot{r}(t) \] and \[ \ddot{p}(t) \]

We can also calculate the energy

\[ E = \frac{1}{2} m \dddot{r}^2 + \frac{1}{2} k \dddot{r}^2 \]

\[ = \frac{1}{2} m \left( \dddot{c}^2 \omega^3 \cos^2 \omega t + \dddot{d}^2 \omega^3 \sin^2 \omega t - 2 \dddot{c} \dddot{d} \omega^3 \sin \omega t \cos \omega t \right) + \]
+ \frac{1}{2} k \left( \bar{c}^2 \sin^2 \omega t + \bar{d}^2 \cos^2 \omega t + 2 \bar{c} \bar{d} \sin \omega t \cos \omega t \right)

= - \bar{c}^2 \left( \frac{1}{2} m \omega^2 \cos^2 \omega t + \frac{1}{2} k \sin^2 \omega t \right) + \bar{d}^2 \left( \frac{1}{2} m \omega^2 \sin^2 \omega t + \frac{1}{2} k \cos^2 \omega t \right) + 2 \bar{c} \bar{d} \left( - \frac{1}{2} m \omega^3 \sin \omega t \cos \omega t + \frac{1}{2} k \sin \omega t \cos \omega t \right)

using \ m \omega^2 = m \left( \frac{k}{m} \right) = k \ gives

= \frac{1}{2} k (\bar{c}^2 + \bar{d}^2) + 0

which is also time independent so both the energy and angular momentum are conserved.

To find the shape it is useful to consider a different pair of independent solutions:

\sin \omega t, \ \cos \omega t \rightarrow \ \sin (\omega t - \phi), \ \cos (\omega t - \phi)

with this choice the general solution has the form

\bar{r}(t) = \bar{a} \cos (\omega t - \phi) + \bar{b} \sin (\omega t - \phi)
to relate these to the previous quantities use

\[
\begin{align*}
\cos (A-B) &= \cos A \cos B + \sin A \sin B \\
\sin (A-B) &= \sin A \cos B - \sin B \cos A
\end{align*}
\]

\[
\tilde{\mathbf{r}}(t) = \tilde{a} \left( \cos \omega t \cos \phi + \sin \omega t \sin \phi \right) + \tilde{b} \left( \sin \omega t \cos \phi - \cos \omega t \sin \phi \right)
\]

\[
= \cos \omega t \left( \tilde{a} \cos \phi - \tilde{b} \sin \phi \right) + \sin \omega t \left( \tilde{a} \sin \phi + \tilde{b} \cos \phi \right)
\]

(note: I used \( \tilde{\mathbf{r}} \) in the book \( \tilde{r} \))

solving \( \mathbf{a} \mathbf{b} \)

\[
\tilde{a} = \cos \phi \tilde{a} + \sin \phi \tilde{b}
\]

\[
\tilde{b} = \cos \phi \tilde{a} - \sin \phi \tilde{b}
\]

we are free to choose \( \phi \) any way that we like - if we choose \( \tilde{a} \cdot \tilde{b} = 0 \)

\[
0 = (\cos^2 \phi - \sin^2 \phi) \tilde{r} \cdot \tilde{d} + \cos \phi \sin \phi \left( c^2 - d^2 \right)
\]

\[
= \cos 2\phi \tilde{r} \cdot \tilde{d} + \frac{1}{2} \sin 2\phi \left( c^2 - d^2 \right)
\]

\[
\sin 2\phi = \cos 2\phi \quad \frac{2c \cdot d}{d^2 - c^2}
\]

\[
\tan 2\phi = \frac{2c \cdot d}{d^2 - c^2}
\]
we can choose axes so \( \mathbf{a} \) and \( \mathbf{b} \) are parallel to the \( x \) and \( y \) axes.

\[
\begin{align*}
x &= a \cos(\omega t - \Phi) \\
y &= b \cos(\omega t - \Phi) \\
z &= 0
\end{align*}
\]

This gives

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2(\omega t - \Phi) + \sin^2(\omega t - \Phi) = 1
\]

which is the equation of an ellipse.

The best way to find \( a \) and \( b \) in terms of \( \mathbf{a} \) and \( \mathbf{b} \) is to use energy and angular momentum conservation

\[
\begin{align*}
E &= \frac{1}{2} k \left( c^2 + d^2 \right) = \frac{1}{2} k \left( \mathbf{a}^2 + \mathbf{b}^2 \right) \\
\mathbf{L} &= \mathbf{a} \times \mathbf{c} \omega = \mathbf{b} \times \mathbf{b} \omega
\end{align*}
\]

Since \( \mathbf{a} \parallel \mathbf{b} \)

\[
\begin{align*}
a b &= c d \sin \alpha \quad (\alpha \text{ angle between } \mathbf{c}, \mathbf{d}) \\
b &= \frac{c d \sin \alpha}{a} \\
c^2 + d^2 &= a^2 + \frac{c^2 d^2 \sin^2 \alpha}{a^2} \\
a^4 &= a^2 (c^2 + d^2) - c^2 d^2 \sin^2 \alpha = 0
\end{align*}
\]
since these equations are completely symmetric with respect to interchanging a, b

The equation is a quadratic equation for a^2. The two roots are a^2 and b^2

\[ a^2, b^2 = \frac{c^2 + d^2 \pm \sqrt{(c^2 + d^2)^2 - 4c^2d^2\sin^2\alpha}}{2} \]

convention

- a^2 = + root (semi major radius)
- b^2 = - root (semi minor radius)

**summary**

1. Motion in a plane
2. Trajectory elliptic
4. Energy, momentum, a, b can all be expressed in terms of \( \vec{e}, \vec{d} \)
5. Note \( \vec{e} \vec{d} = a \vec{e} \vec{b} \) is geometrically a rotation of the coordinates about the Z axis.