Review of mathematics from calculus, linear algebra and differential equations

Complex numbers - complex arithmetic:

\[ i := \sqrt{-1} \]

Complex numbers:
\[ z = z + iy \]

Multiplication of complex numbers:
\[ z_1z_2 = z_2z_1 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \]

Complex conjugation:
\[ z^* = x - iy \]

\[ e^{i\phi} = 1 + \sum_{n=1}^{\infty} \frac{(i\phi)^n}{n!} = \cos(\phi) + i\sin(\phi) \]

Real part of a complex number:
\[ Re(z) = \frac{1}{2}(z + z^*) \]

Imaginary part of a complex number:
\[ Im(z) = -\frac{i}{2}(z - z^*) \]

Exponential representation of a complex number:
\[ re^{i\phi} = r\cos(\phi) + ir\sin(\phi) = x + iy = z \]

Modulus of a complex number:
\[ z^*z = x^2 + y^2 = r^2 \quad |z| = r \]

Argument of a complex number:
\[ \phi; \quad z = re^{i\phi} \]

Inverse of a complex number:
\[ 1/z = z^*/|z|^2 = \frac{x - iy}{x^2 + y^2} = \frac{1}{r}e^{-i\phi} \quad z \neq 0 \]

Natural log of a complex number:
\[ \ln(z) = \ln(r) + i\phi \]
\[ z = e^{\ln(z)} = e^{\ln(r) + i\phi} = re^{i\phi} \]

Complex powers of a complex number:

\[ z_1^{z_2} = e^{\ln(z_1)^{z_2}} = e^{z_2 \ln(z_1)} = e^{z_2 (\ln(r_1) + i\phi_1)} = e^{(x_2 + iy_2)(\ln(r_1) + i\phi_1)} \]

Complex derivatives:

\[ \frac{df}{dz}(z) = \lim_{|z| \to 0} \frac{f(z + re^{i\phi}) - f(z)}{re^{i\phi}} \]

When the complex derivative is defined, it does not depend on \( \phi \! \! \! \! \! \! \). If \( f(z) \) has a complex derivative at \( z \) then \( f(z) \) is called an analytic function at \( z \).

**Taylor’s Theorem:**

\[ \lim_{x \to y} \frac{|f(x) - f(y) - \sum_{n=1}^{N} \frac{1}{n!} \frac{d^n f(y)}{d y^n} (x - y)^n|}{(x - y)^N} = 0 \]

The Taylor expansion of \( f(x) \) about \( y \) is:

\[ f(x) = f(y) + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^n f(y)}{d x^n} (x - y)^n \]

This series does not always converge; when it does converge it does not always converge to \( f(x) \).

**Exponential Taylor series:**

\[ e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \]

converges to \( e^z \) for all complex \( z \). Note that \( 0! := 1 \).

**Geometric Taylor series:**

\[ \frac{1}{1 - z} = 1 + \sum_{n=1}^{\infty} z^n \]
converges to \( \frac{1}{1-z} \) for \( |z| < 1 \)

**Sin and Cosine Taylor series:**

\[
\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad \cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}
\]

converges to \( \sin(z) \) and \( \cos(z) \) for all complex \( z \).

**Ratio test for convergence of series:**

\[
\left| \frac{a_{n+1}z}{a_n} \right| < 1
\]

for sufficiently large \( n \)

\[
\sum a_n z^n
\]

converges.

**Fundamental theorem of algebra:**

Any polynomial of degree \( N \) has \( N \) complex roots:

\[
P(z) = c(z - z_1) \cdots (z - z_N)
\]

where \( c \) is the coefficient of \( z^N \) in \( P(z) \). If \( P(z) \) has real coefficients the roots can still be complex, but they must come in complex conjugate pairs.

**Matrix algebra:**

Matrix components:

\[
A_{ij}
\]

Matrix multiplication:

\[
(AB)_{ij} = \sum_{k=1}^{N} A_{ik} B_{kj} \quad AB \neq BA
\]

Matrix transpose:

\[
A^t_{ij} = A_{ji}
\]

Complex conjugation:

\[
A^*_{ij}
\]

Matrix-adjoint:

\[
A^\dagger_{ij} = A^*_{ji}
\]
Exponential of matrix:

\[ e^A = I + \sum_{n=1}^{\infty} \frac{1}{n!} A^n \]

converges for any \( A \) with \( |A_{ij}| < \infty \)

Determinant of a matrix:

\[ \det(A) = \sum_{\mu_1 \cdots \mu_N} \epsilon_{\mu_1 \cdots \mu_N} \prod_{k=1}^{N} A_{\mu_k k} \]

where \( \epsilon_{\mu_1 \cdots \mu_N} \) is completely anti-symmetric and normalized by

\[ \epsilon_{1,2 \cdots N} = 1 \]

It can also be defined by

\[ \det(A) = (-)^{i+1} A_{i1} \det(\hat{A}_{i1}) + (-)^{i+2} A_{i2} \det(\hat{A}_{i2}) + \cdots + (-)^{i+N} A_{iN} \det(\hat{A}_{iN}) \]

where \( \hat{A}_{ij} \) is the submatrix obtained from \( A \) by removing the \( i \)-th row and \( j \)-th column from \( A \).

Product of determinants:

\[ \det(AB) = \det(A)\det(B) = \det(BA) \]

Matrix inverse:

\[ A^{-1}_{ij} = (-)^{i+j} \frac{\det(\hat{A}_{ij})}{\det(A)} \]

Existence requires \( \det(A) \neq 0 \).

Solution of linear equations:

\[ \sum_{j=1}^{N} A_{ij} x_j = b_i \quad x_i = \sum_{j=1}^{N} A^{-1}_{ij} b_j \]

can be solved if and only if \( \det(A) \neq 0 \)

Eigenvalues and eigenvectors:

\[ \sum_{j=1}^{N} (A_{ij} - \lambda \delta_{ij}) v_j = 0 \]

\( \lambda \) satisfying the above is called an eigenvalue. \( \mathbf{v} = (v_1, v_2, \cdots, v_N) \) is the eigenvector associated with \( \lambda \).

A non-zero \( \mathbf{v} \) requires

\[ \det((A_{ij} - \lambda \delta_{ij}) = P(\lambda) = 0 \]

otherwise the matrix has an inverse - leading to \( \mathbf{v} = 0 \).
This equation is a polynomial of degree \( N \) in \( \lambda \). It has \( N \) roots by the fundamental theorem of algebra. This means that there are \( N \) eigenvalues. If the roots are all different there are \( N \) eigenvectors:

\[
A v_m = \lambda_m v_m
\]

A general vector can be written as

\[
v = \sum_{n=1}^{N} c_n v_n
\]

Cayley-Hamilton theorem:

\[
P(A) = c \prod_{m=1}^{N} (A - \lambda_m) = 0.
\]

If the eigenvalues are all different

\[
\Pi_k = \prod_{j \neq k} \frac{(A - \lambda_j)}{(\lambda_k - \lambda_j)}
\]

is a projection on the \( j \)-th eigenvector

\[
\Pi_j \Pi_i = \delta_{ij} \Pi_i
\]

\[
\Pi_j v_k = \delta_{ik} v_k
\]

\[
\Pi_j v = \Pi_j (\sum_{n=1}^{N} c_n v_n) = c_j v_j
\]

If \( v = 0 \) then \( c_j v_j = 0 \) which means that \( c_j = 0 \). This implies that the \( N \) eigenvectors are independent. This leads to

\[
I = \sum_{j=1}^{N} \Pi_j
\]

and

\[
A = \sum_{j=1}^{N} \lambda_j P_j
\]

When the eigenvectors are not all different these results have to be modified.

Functions of matrices

\[
f(A) = \sum_{j=1}^{N} f(\lambda_j) \Pi_j
\]

Hermitian matrices: \( A = A^\dagger \).

Unitary matrices: \( AA^\dagger = A^\dagger A = I \).
Normal matrices: \([A, A^\dagger] = 0\).
Hermitian and unitary matrices are normal. The eigenvectors of normal matrices can always be chosen to be orthogonal.

**Binomial series:**

\[(x + y)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k y^{n-k}\]

**Multinomial series:**

\[(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1, \ldots, k_m | \sum_k=k} \frac{n!}{k_1! \cdots k_m!} x_1^{k_1} \cdots x_m^{k_m}\]

**Trigonometry identities**

\[
\begin{align*}
\sin(a \pm b) &= \sin(a) \cos(b) \pm \cos(a) \sin(b) \\
\cos(a \pm b) &= \cos(a) \cos(b) \mp \sin(a) \sin(b) \\
\sin(2a) &= 2 \sin(a) \cos(a) \\
\cos(2a) &= \cos^2(a) - \sin^2(a) = 2 \cos^2(a) - 1 \\
\sin\left(\frac{a}{2}\right) &= \pm \sqrt{\frac{1 - \cos(a)}{2}} \\
\cos\left(\frac{a}{2}\right) &= \pm \sqrt{\frac{1 + \cos(a)}{2}} \\
\sin(a) &= \frac{e^{ia} - e^{-ia}}{2i} \\
\cos(a) &= \frac{e^{ia} + e^{-ia}}{2}
\end{align*}
\]

**Differential equations:**

n-th order differential equation:

\[
\frac{d^n f}{dx^n} + \sum_{k=0}^{n-1} a_k(x) \frac{d^k f}{dx^k} = 0
\]

This simplest way to understand how differential equations work is to note \(\frac{d^{n+k} f}{dx^{n+k}}\) can be generated by differentiating the differential equation \(k\) times. The
Taylor expansion of the solution can be constructed by evaluating all of these derivatives at the initial point:

\[ f(x) = f(x_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^n f(x_0)}{dx^n} (x - x_0)^n \]

This may or may not converge. The following integral method due to Picard converges for sufficiently small \((x - x_0)\) provided the coefficients \(a_k(x)\) are continuously differentiable.

Picard’s method:
To apply this method the first step is to reduce the n-th order differential equation to an equivalent system of n first order equations:

\[
\begin{align*}
\frac{df}{dx} &= y_1(x) \\
\frac{dy_1}{dx} &= y_2(x) \\
&\vdots \\
\frac{dy_k}{dx} &= y_{k+1}(x) \\
\frac{dy_{n-1}}{dx} &= -\sum_{k=0}^{n-1} a_k(x)y_k(x)
\end{align*}
\]

This has the form

\[
\frac{dy}{dx} = g(y(x), x)
\]

where \(y_0(x) = f(x)\). This can be approximately solved by taking sufficiently small steps

\[
x_n = x_0 + n\Delta x \\
y(x_{n+1}) \approx y(x_n) + g(y(x_n), x_n)\Delta x.
\]

Alternatively the differential equation can be converted to an integral equation

\[ y(x) = y(x_0) + \int_{x_0}^{x} g(y(x'), x')dx' \]

which can be solved by iteration

\[
\begin{align*}
y(x) &= \lim_{n \to \infty} y_n(x) \\
y_1(x) &= y(x_0) \\
y_{n+1}(x) &= y(x_0) + \int_{x_0}^{x} g(y_n(x'), x')dx'
\end{align*}
\]
This converges for sufficiently small \( x - x_0 \) provided \( g \) has continuous derivatives. The equation

\[
\frac{dy}{dx} = g(y(x), x)
\]

can be used to generate a graphical picture of the solution starting from some initial point.

\( N \)th order linear differential equations with constant coefficients can be reduced to a system of \( N \) first order equations of the form

\[
\frac{dy}{dx} = Ay
\]

where \( A \) is a constant matrix. Differentiating \( n \)-times gives

\[
\frac{d^n y}{dx^n} = A^n y
\]

where \( A^n \) is the matrix product of \( A \) \( n \) times. This can be used to construct the Taylor series for the solution

\[
y(x) = y(x_0) + \sum_{n=1}^{\infty} \frac{1}{n!} A^n(x - x_0)^n.
\]

This series converges for any matrix \( A \). This is written as

\[
y(x) = e^{A(x-x_0)}y(x_0)
\]

Vector Calculus:

Partial derivatives depend on which variable is held constant. Consider \( f(x, y, z) \). Change variables \( (x, y, z) \rightarrow (x, a, b) \) where \( a \) and \( b \) are functions of \( (x, y, z) \). Then in general

\[
\frac{\partial f(x, y, z)}{\partial x} \bigg|_{y, z} \neq \frac{\partial f(x, a, b)}{\partial x} \bigg|_{a, b}
\]

\( f(x_1 \cdots x_n) \) has a stationary point at \( x = (x_1, \cdots, x_n) \) if

\[
\frac{\partial f(x)}{\partial x_i} \bigg|_{x_1, \cdots, x_i} = 0 \quad \text{for all } i.
\]

\( f(x_1, \cdots, x_n) \) has a local minimum at \( x = (x_1, \cdots, x_n) \) if \( x \) is a stationary point of \( f(x) \) and

\[
\frac{\partial^2 f(x)}{\partial x_i \partial x_j}
\]

is a matrix with \( n \) positive eigenvalues.

\( f(x_1, \cdots, x_n) \) has a local maximum at \( x = (x_1, \cdots, x_n) \) if \( x \) is a stationary point of \( f(x) \) and

\[
\frac{\partial^2 f(x)}{\partial x_i \partial x_j}
\]

is a matrix with \( n \) negative eigenvalues.