Lecture 34

For a field \( E = E_0 \hat{x} \cos \omega t \) \( V = - q E_0 x \cos \omega t \)

\[
P_{i \rightarrow j} = E_0^2 |\Psi_i| \times |\Psi_j|^2 \frac{\sin^2 \left( \frac{(\omega_i - \omega_j + \omega) t}{2} \right)}{\kappa^2 (\omega_i - \omega_j + \omega)^2}
\]

\( \omega_i > \omega_j \Rightarrow 0 - \omega \)

\( \omega_j > \omega_i \Rightarrow \omega + \omega \)

This probability is sharply peaked when \( \omega = \omega_i - \omega_j \) \( \omega_j - \omega_i = - \omega \)

depending on the sign of \( E_i - E_j \).

This is the probability of the lower state being excited to the upper state or stimulated emission of the upper state to the lower state.

The interesting observation is that both quantities are the same.

The other kind of emission is spontaneous decay of the upper state. If it was a true bound state it would never decay, but in general the system is not isolated - it interacts.
with its environment. In general the interactions involve electromagnetic radiation rather than the weak, strong, or gravitational forces.

The electromagnetic energy density for \( \mathbf{E} = E_0 \hat{x} \cos(\omega t) \) is

\[
U_E = \frac{E_0^2}{2} \cos^2(\omega t)
\]

for an electromagnetic wave there is also a magnetic component with energy density

\[
U_B = \frac{1}{2\mu_0} B_0^2 \cos^2(\omega t)
\]

but \( B_0 = E_0/c \quad E_0 \mu_0 = \frac{1}{c^2} \quad C = speed \ of \ light \)

\[
U_T = E_0 E_0^2 \cos^2(\omega t)
\]

the time average

\[
\frac{1}{T} \int_0^T \cos^2(\omega t) dt = \frac{1}{2} \quad (T = \frac{2\pi}{\omega})
\]

this gives

\[
E_0^2 = 2 \frac{\langle U_T \rangle}{E_0^2} \quad \langle U \rangle = E_0 E_0^2 \frac{1}{2}
\]
which gives the expression

\[ p(\omega) = \frac{\langle u \rangle}{\epsilon_0 \hbar} |\langle \psi_i | \hat{r} \cdot \hat{x} | \psi_j \rangle|^2 \left( \frac{\sin^2 \left( \frac{\omega_i - \omega_j}{2} \right)}{(\omega_i - \omega_j) \omega} \right)^2 \]

where \( \hat{x} \) is the direction of the polarization of the electric field.

In general the atom will be in an area with many frequencies, directions of propagation and polarizations

\[ dp(\omega) = \frac{p(\omega) d\omega}{\epsilon_0 \hbar^2} |\langle \psi_i | \hat{r} \cdot \hat{x} \bar{\psi} | \psi_j \rangle|^2 \left( \frac{\sin^2 \left( \frac{\omega_i - \omega_j}{2} \right)}{(\omega_i - \omega_j) \omega} \right)^2 \]

the total probability for a transition from \( i \) to \( j \) is obtained by integrating this over \( \omega \)

\[ p_{ji} = \int \frac{p(\omega) d\omega}{\epsilon_0 \hbar^2} |\langle \psi_i | \hat{r} \cdot \hat{x} | \psi_j \rangle|^2 \left( \frac{\sin^2 \left( \frac{\omega_i - \omega_j}{2} \right)}{(\omega_i - \omega_j) \omega} \right)^2 \]

This is sharply peaked near \( \omega_i - \omega_j \) on \( \omega_j - \omega_i \).
We assume that it is a good approximation to neglect this outside of the integral

\[ p \approx p(\omega_i - \omega_j) \int_0^\infty \frac{d\omega}{e^{\omega^2}} \frac{K t_1 \Gamma X Y X Y}{(\omega - \omega_i)^2} \]

Again since this is sharply peaked, we extend the integral from \(-\infty\) to \(\infty\),

\[ p = p(\omega_i - \omega_j) \int_{-\infty}^{\infty} \frac{d\omega}{e^{\omega^2}} K t_1 \Gamma X Y X Y / (\omega - \omega_i)^2 \sin^2 \left( \frac{\omega - \omega_i + \omega_j}{2} \right) \]

Let \[ x = (\omega_i - \omega_j + \omega) \]

\[ dx = \frac{d\omega}{2e^{\omega^2}} \quad d\omega = \frac{2}{e^{\omega^2}} dx \]

\[ p(\omega_i - \omega_j) \int_{-\infty}^{\infty} \frac{2}{e^{\omega^2}} K t_1 \Gamma X Y X Y / (\omega - \omega_i)^2 \sin^2 x \]

The integral

\[ \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} \, dx = \pi \]

This gives the final expression for the probability

\[ p(\omega_i - \omega_j) \frac{1}{e^{\omega^2}} K t_1 \Gamma X Y X Y \]
1. recall the same value of $B_0$ was responsible for $\pm \omega$ so
   \[ \rho(\omega_1,\omega_2) = \rho(\omega_1,\omega_2) \]

2. This is valid in first order perturbation theory - when this is valid the transition probability is proportional to $t$
   
   The transition rate \[ \frac{dp}{dt} = R = \rho(\omega) \langle \mathcal{F}_i \mid \mathcal{F}_i \rangle \]
   is constant \[ R_{ii} = R_{ii} \]

3. $\mathbf{x}$ is the direction of polarization of the electric field - it is perpendicular to the line direction of propagation of the radiation.

In unpolarized radiation in arbitrary directions all polarizations are equally probable

To average over polarizations let
   \[ \mathbf{v} = \langle \mathcal{F}_i \mid \mathcal{F}_i \rangle \]
   and choose coordinates so $\mathbf{v}$ is in the $\hat{z}$ direction

   \[ \langle \mathbf{v}, \hat{z}, \mathbf{v}, \hat{z} \rangle = \frac{\int v^2 \cos^2 \theta \ dS}{\int dS} = \frac{\int v^2 \frac{1}{4\pi} \ dS}{\int dS} \]
   \[ = \frac{\int v^2 \frac{1}{4\pi} \ dS}{\frac{1}{3} + \frac{1}{3}} = \frac{\int v^2 \frac{1}{4\pi} \ dS}{3} \]
The resulting transition rate for excitation on stimulated emission due to unpolarized radiation in arbitrary directions is

$$R = P(\omega) \frac{\langle \psi_i | \vec{E} | \psi_f \rangle \cdot \langle \psi_f | \vec{E} | \psi_i \rangle}{3 e_0 \hbar^2}$$

Selection rules determine when a transition is possible - if

$$\langle \psi_i | \vec{E} | \psi_f \rangle = 0$$

no first order transition is possible.

Cases

1. $|\psi_i\rangle$ $|$ $|\psi_f\rangle$ states of the same parity

2. $|\psi_i - \psi_f\rangle$ $|$ due to Clebsch-Gordan coefficients

Recall $\tilde{F} \sim F \times \text{linear combination}$ of $\gamma_i^m$

$$\gamma_i^m = \sqrt{\frac{3}{2\pi}} \frac{m}{F}$$

$$\gamma_i^1 = -\sqrt{\frac{3}{2\pi}} (x + iy)$$

$$\gamma_i^{-1} = \sqrt{\frac{3}{2\pi}} (x - iy)$$
This analysis does not account for spontaneous emission

assume $E_b > E_a$ and there are $N = N_a + N_b$ atoms; $N_a$ in state $a$ and $N_b$ in state $b$.

Cusanus

$$\frac{dN_b}{dt} = -N_b R_{ba} - N_b A + N_a R_{ab}$$

where $R_{ba} = \rho(\omega_{ba}) B_{ba} \quad R_{ab} = \rho(\omega_{ab}) B_{ab}$

$$= \rho(\omega_{ba}) (N_a B_{ba} - N_b B_{ab}) - N_b A$$

In thermal equilibrium $\frac{dN_b}{dt} = 0$

then we can solve for $\rho(\omega_{ba})$

$$\rho(\omega_{ba}) = \frac{N_b A}{N_a B_{ba} - N_b B_{ab}} = \frac{A}{\frac{N_a}{N_b} B_{ba} - B_{ab}}$$

In thermal equilibrium at temperature $T$ (in Kelvin)
\[
\frac{N_a}{N_0} = \frac{e^{-E_a/kT}}{e^{-E_b/kT}} = e^{\frac{(E_b - E_a)/kT}{\epsilon \hbar \omega_0 / kT}} = e
\]

\[
\rho(\omega n) = \frac{A}{e^{\frac{\hbar \omega_0 / kT}{\epsilon}} \left( B_{en} - B_{bo} \right)}
\]

Einstein compared this to Planck blackbody radiation law:

\[
\rho(\omega) = \frac{\hbar \omega^3}{\pi^2 c^3 \left( e^{\frac{\hbar \omega / kT}{\epsilon}} - 1 \right)}
\]

The only way to reconcile these formulas was to have

1. \( B_{en} = B_{bo} \) which we already derived

2. \( \frac{A}{B_{en}} = \frac{\hbar \omega^3}{\pi^2 c^3} \)

\[
A = \frac{\hbar \omega^3}{\pi c^3}, \quad B_{en} = \frac{\hbar \omega^3}{\pi c^3}, \quad \frac{K \epsilon_0 \hbar / 4 \pi}{3 e^2 c^2}
\]

\[
A = \frac{c \omega^3}{3 \pi \epsilon_0 \hbar c^3} <\psi_{en} | \psi_{en}>^2
\]
Thus, using the blackbody radiation result we conclude that the transition rate for spontaneous emission is $A$

If we have $N_0$ atoms in the absence of an applied electric field

$$\frac{dN_b}{dt} = -AN_b$$

which gives

$$N_a(t) = N_a(0)e^{-At} = N_a e^{-t/\tau}$$

$$\gamma = \frac{1}{\tau} = \frac{3\pi\varepsilon_0 e^4}{m^3 c^3} \frac{1}{K_{ab}(\omega^2 \tau_1)}$$

where $\tau$ is the lifetime of the excited state for decay into state $b$

If there are $c$ number of possible decay states

$$\frac{dN_b}{dt} = \sum_{c} A_{ab}N_b$$

$$\frac{1}{\tau_b} = \frac{1}{\tau_0} + \frac{1}{\tau_{100}}$$
So far we have only considered decay from one eigenstate of $H_0$ to another eigenstate of $H_0$.

If the atom is in a state $|\psi_i\rangle$ of $H_0$ and the electric field is strong enough to ionize the atom then $E_f = E_i + \hbar \omega$; in this case the final state involves the ion and a free electron, which has a variable relativistic momentum. In this case $E_f$ is not discrete. Instead there is a density of final states

$$dE_f = \rho(E) dE$$

In this case we can ask for the probability of finding a transition to a state with energy between $E_f - \Delta E_f/2$ to $E_f + \Delta E_f/2$

$$P = \int_{E_f - \Delta E_f/2}^{E_f + \Delta E_f/2} \frac{K |E| V |E| \beta}{(E_f - E_i - \hbar \omega)^2} \sin^2 \left( \frac{E_f - E_i - \hbar \omega}{2} \right) \rho(E) dE$$
Because this is sharply peaked for large $t$ we can take $\rho(E)KE_{i}V_{i}E_{i}^{2}$ out of the integrals

$$P = \rho(E_{i})KE_{i}V_{i}E_{i}^{2} \int_{E_{i}-\Delta E}^{E_{i}+\Delta E} \frac{\sin^{2} \left( \frac{E_{j}-E_{i}+E}{2k} \right)}{(E_{j}-E_{i}-E)^{2}} dE$$

We also let the limits of integration go from $-\infty$ to $\infty$

$$P = \rho(E_{i})KE_{i}V_{i}E_{i}^{2} \int_{-\infty}^{\infty} \frac{\sin^{2} \left( \frac{E_{j}-E_{i}+E}{2k} \right)}{(E_{j}-E_{i}-E)^{2}} dE$$

Let $x = \frac{(E_{j}-E_{i}+E)t}{2k}$, so $\frac{2k}{t} dx = -dE$

$$= \rho(E_{i}) KE_{i}V_{i}E_{i}^{2} \int_{-\infty}^{\infty} \frac{\sin^{2} x}{(E_{j}-E_{i}-E)^{2}} \frac{2k}{t} dx$$

$$= \rho(E_{i}) KE_{i}V_{i}E_{i}^{2} \frac{t}{2k} \pi$$

differentiating with respect to $t$ gives the rate of transitions to states with final energy $E_{j}$

$$R = \rho(E) \frac{t}{2k} KE_{j}V_{j}E_{j}^{2}$$
as in the case with incoherent radiation, the transition rate for long times is constant (this assumes that first order perturbation theory is valid).

To use this, we need to calculate the density of final states, since they can have any momentum $p(E) \, dE$.

In a free particle, $E = \frac{p^2}{2m}$, $\frac{dE}{dp} = \frac{p}{m}$, $\frac{dp}{dE} = \frac{m}{p}$.

$dp\,dP = p^2 \, dp \sin\theta \, d\theta \, d\phi$

$= p^2 \frac{dp}{dE} \sin\theta \, d\theta \, d\phi \, dE$

If we integrate over angle:

$= p^2 \frac{m}{p} \, 4\pi \, dE$

$= 4\pi mp \, dE$

In this case, $p(E) = 4\pi mp$ if the particles can have 2 spin states as well then this must be doubled

$= 8\pi mp \, dE$
application to scattering - 

in this case the interaction is \( V \) rather than \( V \cos \omega t \). The factor \( 1/2 \) leads to a factor of 4 in the denominator of \( R \)

\[
R_{\text{scatt}} v \rightarrow \rho(E) \frac{2\pi}{h} |K_{P+1} V_{P+1}|^2
\]

the differential cross section is the ratio of the transition rate to the incident current, i.e.

\[
\langle \bar{r} | \rho \rangle = \frac{1}{(2\pi\hbar)^3} \int \rho_{\bar{r},\bar{r}} \bar{r} d\bar{r} = \psi^*(x) (-i\hbar \nabla / m) \psi(x) = \frac{\overline{P}}{m} \frac{1}{2\pi \hbar^2}
\]

we also have

\[
K_{P+1} V_{P+1} \rightarrow \frac{1}{(2\pi\hbar)^3} \int \rho_{\bar{r},\bar{r}} \bar{r} d\bar{r} = \frac{\rho}{\hbar} \left[ \sum \int \rho_{\bar{r},\bar{r}} \bar{r} d\bar{r} \right] \left[ \int \rho_{\bar{r},\bar{r}} \bar{r} d\bar{r} \right] \left[ \int \rho_{\bar{r},\bar{r}} \bar{r} d\bar{r} \right]
\]

\[
d\sigma = \frac{d\rho}{d\Omega} = \frac{2\pi}{\hbar} \left( \frac{2\pi \hbar)^3}{2\pi \hbar^2} \right) m \left[ \int \rho_{\bar{r},\bar{r}} \bar{r} d\bar{r} \right] \left[ \int \rho_{\bar{r},\bar{r}} \bar{r} d\bar{r} \right] \left[ \int \rho_{\bar{r},\bar{r}} \bar{r} d\bar{r} \right]
\]

where \( \frac{d\sigma}{d\Omega} = m \rho \rightarrow \)

\[
d\sigma = \frac{m^2}{\hbar^2 4\pi^2} \left[ \int \rho_{\bar{r},\bar{r}} \bar{r} d\bar{r} \right] \left[ \int \rho_{\bar{r},\bar{r}} \bar{r} d\bar{r} \right] \left[ \int \rho_{\bar{r},\bar{r}} \bar{r} d\bar{r} \right]
\]

\[
= \frac{m^2}{2\pi \hbar^2} \left[ \int \rho_{\bar{r},\bar{r}} \bar{r} d\bar{r} \right] \left[ \int \rho_{\bar{r},\bar{r}} \bar{r} d\bar{r} \right] \left[ \int \rho_{\bar{r},\bar{r}} \bar{r} d\bar{r} \right]
\]

is the Born approximation.
Adiabatic Theorem

Consider a Hamiltonian that has parameters that depend on time.

Example

\[ H(t) = -\frac{k}{2m} \frac{d^2}{dt^2} + \frac{1}{2} m \omega^2(t)x^2 \]

This is a harmonic oscillator with a time-varying frequency.

For each fixed time there are eigenvalues and eigenvalues:

\[ E_n = \hbar \omega (n(t) + \frac{1}{2}) \]

\[ \langle \psi | \psi_n(t) \rangle \]

In general for 2 different times there are different Hamiltonians and while the wave functions in both cases are normalized; the phases are arbitrary.
on the other hand if we find a set of solutions \( \psi(t=0) \) that are well defined functions of \( \omega \), then as \( t \) is varied the phase becomes a function of \( t \).

The statement of the theorem is

\[
\psi(t) = \sum C_n(\omega) |\phi_n(\omega)\rangle
\]

\[
\psi(t) = \sum C_n(t) |\phi_n(t)\rangle
\]

Where

\[
C_n(t) = e^{-i \int_0^t E_n(t')dt'} - i \int_0^t \langle n|\hat{A}(t')|n\rangle dt'
\]

The first term is called the dynamical phase, while the second term is called the geometric phase.

While in general the phase is not observable, if \( H(t) \) returns to its original value, the change in the geometric phase is an observable quantity — called the Berry phase.
This can be checked by differentiating
\[ i \hbar \frac{d}{dt} \left( \sum_n C_n(t) e^{i \int_0^t \hat{V}(t') dt'} - \sum_n \langle \hat{\theta}(t') | \hat{n}(t) \rangle \right) \]
\[ = \sum_n C_n(t) \left( (E_n + i \hbar \langle \hat{\theta}(t) | \hat{n}(t) \rangle) e^{i \int_0^t \hat{V}(t') dt'} - \sum_n \langle \hat{\theta}(t') | \hat{n}(t) \rangle \right) \]
\[ = \sum_n C_n(t) \left( (E_n - E_n - E_n) e^{i \int_0^t \hat{V}(t') dt'} - \sum_n \langle \hat{\theta}(t') | \hat{n}(t) \rangle \right) \]
\[ = \sum_n C_n(t) E_n(t) e^{i \int_0^t \hat{V}(t') dt'} - \sum_n \langle \hat{\theta}(t') | \hat{n}(t) \rangle \]

when this can break down is
when there are level crossings —
— i.e. when 2 states have the
same energy at time \( t \).