1. (WKB approximation) Consider a particle of mass \( m \) in a harmonic oscillator well, \( V(x) = \frac{1}{2}kx^2 \). Use units where \( m = 1, \ k = 1 \).

a. What is the classical momentum for this particle?
b. What are the classical turning points for this system?
c. What is the form of the WKB wave function in the well?
d. What is the WKB quantization condition?
e. Find the approximate eigenvalues in the WKB approximation?

2. (Scattering) The Hamiltonian for two particles of mass \( m \) interacting with an exponential potential

\[
V(r) = -\lambda e^{-ar^2}
\]

is

\[
h = \frac{p^2}{2\mu} + V(r)
\]

where \( \mu = \frac{m}{2} \) is the reduced mass and \( p \) is the relative momentum.

a. Find the scattering amplitude for this reaction in the first Born approximation.
b. Find the differential cross section in the first Born approximation.
c. Find the total cross section in the first Born approximation.

3. (Time dependent perturbation theory)

a. A two state system has Hamiltonian

\[
H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}
\]

At time \( t = 0 \) it experiences an interaction of the form

\[
V = \lambda \theta(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cos(\omega t)
\]

a. Write down the eigenvectors corresponding to eigenvalues \( E_1 \) and \( E_2 \).
b. Use first order perturbation theory find the probability for transition from state 1 to state 2 as a function of time.
c. At what angular frequency is this probability maximal?
d. What is the transition probability for a transition from state 2 to state 1 as a function of time?

\[
\int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\frac{\pi}{a}} \quad \text{real}(a) \geq 0
\]
Solutions

1.a Classical momentum:

\[ p_{cl}(x) = \sqrt{2(E - \frac{1}{2}x^2)} \]

1.b Classical turning points:

\[ x_t = \pm \sqrt{2E} \]

1.c WKB wave function (in well):

\[ \psi(x) = \frac{N}{\sqrt{p_{cl}(x)}} \cos\left(\frac{1}{\hbar} \int p_{cl}(x')dx' + \phi\right) \]

1.d WKB quantization condition:

\[ \int_{x_-}^{x_+} p_{cl}(x)dx = \pi \hbar (n + \frac{1}{2}) \]

1.d WKB energy eigenvalues:

\[ \int_{x_-}^{x_+} p_{cl}(x)dx = \pi \hbar (n + \frac{1}{2}) \]

\[ \pi \hbar (n + \frac{1}{2}) = \int_{-\sqrt{2E}}^{\sqrt{2E}} dx \sqrt{2(E - \frac{1}{2}x^2)} \]

\[ u = x\sqrt{\frac{1}{2E}} \]

\[ \pi \hbar (n + \frac{1}{2}) = 2E \int_{-1}^{1} du \sqrt{1 - u^2} = 2E \int_{0}^{\pi} d\phi \sin^2(\phi) = E\pi \]

where \( u = \cos(\phi) \)

\[ E_n = \hbar(n + \frac{1}{2}) \]

2.a Scattering amplitude:

\[ F = -\frac{\mu}{2\pi \hbar^2} \int e^{i\mathbf{r} \cdot (\mathbf{p}' - \mathbf{p})/\hbar (\mathbf{p}' - \mathbf{p})^2/2\hbar^2} \mathbf{x} \]

complete the square and write as the product of three one-dimensional integrals. Use the formula at the bottom of the exam to do the Gaussian integral.

\[ F = -\frac{\mu}{2\pi \hbar^2} (\mathbf{r})^3/2 \left( \int e^{\alpha(x'_i - \frac{(p'_i - p_i)^2}{2m^2})^2 - \frac{(p'_i - p_i)^2}{4\hbar^2} dx'_i} \right) = \]

\[ \frac{\mu \lambda}{2\pi \hbar^2} \frac{\pi}{\alpha} 3/2 e^{-\frac{(p'_i - p_i)^2}{4\hbar^2}} = \frac{\mu \lambda}{2\pi \hbar^2} \frac{\pi}{\alpha} 3/2 e^{-\frac{1}{2\hbar^2} \cos(\theta)} \]
2.b Differential cross section:

\[ d\sigma = F^2 d\Omega = \frac{\mu^2 \lambda^2}{4\pi^2 \hbar^4} \left( \frac{\pi}{\alpha} \right)^3 e^{-\frac{2\sqrt{(1 - \cos(\theta))}}{\hbar \alpha}} \sin(\theta) d\theta d\phi \]

2.c Total cross section. Let \( u = \cos(\theta) \)

\[ \sigma = \int_0^{2\pi} \int_0^\pi \sin(\theta) d\theta \frac{\mu^2 \lambda^2}{4\pi^2 \hbar^4} \left( \frac{\pi}{\alpha} \right)^3 e^{-\frac{2\sqrt{(1 - u)}}{\hbar \alpha}} = \]

\[ \int_{-1}^1 du \frac{2\pi \mu^2 \lambda^2}{4\pi^2 \hbar^4} \left( \frac{\pi}{\alpha} \right)^3 e^{-\frac{2\sqrt{(1 - u)}}{\hbar \alpha}} = \frac{2\hbar^2 \alpha \pi \mu^2 \lambda^2}{4\pi^2 \hbar^4 p^2} (1 - e^{-\frac{2\sqrt{u}}{\hbar \alpha}}) = \]

\[ \frac{\alpha \mu^2 \lambda^2}{\pi \hbar^2} e^{-\frac{p^2}{\hbar^2 \alpha}} \frac{\sinh^2 \left( \frac{p^2}{\hbar^2 \alpha} \right)}{p^2} \]

3.a Eigenvectors:

\[ \psi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \psi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

3.b Assume the particle is initially in state 1

\[ P = \lambda^2 \langle \psi_1 | \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} | \psi_2 \rangle^2 \frac{\sin^2 \left( \frac{(E_2 - E_1 - \hbar \omega)t}{2\hbar} \right)}{(E_2 - E_1 - \hbar \omega)^2} = \]

\[ \lambda^2 \frac{\sin^2 \left( \frac{(E_2 - E_1 - \hbar \omega)t}{2\hbar} \right)}{(E_2 - E_1 - \hbar \omega)^2} \]

3.c \( \omega = (E_2 - E_1)/\hbar \)

3.d Same as part b.

\[ P = \lambda^2 \frac{\sin^2 \left( \frac{(E_2 - \hbar \omega - E_1)t}{2\hbar} \right)}{(E_2 - \hbar \omega - E_1)^2} \]