Lecture 1 - 1/19/22

Class Organization

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  - The class web page will have posted lectures, posted homework assignments, homework solutions after they are due, sample exams, and general messages. It can be reached using the address above or on the first page of the class ICON site or on my faculty page. I will not be using ICON for anything else.

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- Grades: There will be two mid terms and a final. In addition there will be weekly homework assignments except before midterms. They count 25% for each mid term, 35% for the final exam, 15% for the homework. You will be allowed 1 side of an 8 1/2 x 11 sheet of notes for each mid term and 2 sheets for the final. All exams will be curved and scaled so the mean is 70 and the standard deviation is 15 before adding up the total score. Possible course grades are A+, A, B, C, D, F, with an A+ being a full grade above an A. While the homework counts only 15% it is essential if you want to learn quantum mechanics.

- Questions are encouraged and will be appreciated by your peers. Quantum mechanics is not intuitive and question help alleviate confusion.
Chapter 5: Identical Particles

The first few lectures will be on chapter 5 in the text. It is on the quantum mechanics of systems of identical particles.

The wave function for a single spinless particle is a function of the particle’s position and time:

\[ \Psi(r, t) \]

If the wave function is normalized to 1

\[ \int \Psi^*(r, t)\Psi(r, t)d^3r = 1 \]

then

\[ dP(r, t) := \Psi^*(r, t)\Psi(r, t)d^3r \]

represents the differential probability of finding the particle within \( d^3r \) of \( r \) at time \( t \).

If the wave function is not normalized to 1 then this is replaced by

\[ dP(r, t) := \frac{\Psi^*(r, t)\Psi(r, t)d^3r}{\int \Psi^*(r, t)\Psi(r, t)d^3r} \]

Although \( dP(r, t) \) will change as a function of \( t \), i.e. there will be times when the differential probability of finding a particle near a particular point will be small, but the total probability, which represents the probability of finding the particle somewhere, will remain 1 independent of time.

The wave function for 2 particles is a function of the coordinates of each particle and the time

\[ \Psi(r_1, r_2, t) \]

As in the one particle case it is convenient to assume that the wave function is normalized to 1

\[ \int \Psi^*(r_1, r_2, t)\Psi(r_1, r_2, t)d^3r_1d^3r_2 = 1 \]

In this case

\[ dP(r_2, r_2, t) := \Psi^*(r_1, r_2, t)\Psi(r_1, r_2, t)d^3r_1d^3r_2 \]

represents the probability of finding particle 1 within \( d^3r_1 \) of \( r_1 \) and particle 2 within \( d^3r_2 \) of \( r_2 \) at time \( t \).
The time dependence of the wave function is determined by solving the Schrödinger equation using the Hamiltonian of the two particle system with some initial condition

$$i\hbar \frac{\partial \Psi(r_1, r_2, t)}{\partial t} = \left( -\frac{\hbar^2}{2m_1} \nabla^2_{r_1} - \frac{\hbar^2}{2m_2} \nabla^2_{r_2} + V(r_1, r_2, t) \right) \Psi(r_1, r_2, t)$$

with

$$\Psi(r_1, r_2, 0)$$

as the initial condition. The operator

$$H := \left( -\frac{\hbar^2}{2m_1} \nabla^2_{r_1} - \frac{\hbar^2}{2m_2} \nabla^2_{r_2} + V(r_1, r_2, t) \right)$$

is the **Hamiltonian** of the two particle system.

The Schrödinger equation is a linear partial differential equation. In general it must be solved numerically. Some simplifications are discussed below.

**Time-independent potentials**

In this case we try to find solutions of the form

$$\Psi(r_1, r_2, t) = \Psi_a(r_1, r_2)\Psi_b(t)$$

Using this in the Schrödinger equation gives

$$i\hbar \Psi_a(r_1, r_2) \frac{\partial \Psi_b(t)}{\partial t} = \Psi_b(t) \left( -\frac{\hbar^2}{2m_1} \nabla^2_{r_1} - \frac{\hbar^2}{2m_2} \nabla^2_{r_2} + V(r_1, r_2) \right) \Psi_a(r_1, r_2)$$

Dividing both sided by $\Psi_a(r_1, r_2)\Psi_b(t)$ gives

$$i\hbar \frac{1}{\Psi_b(t)} \frac{\partial \Psi_b(t)}{\partial t} = \frac{1}{\Psi_a(r_1, r_2)} \left( -\frac{\hbar^2}{2m_1} \nabla^2_{r_1} - \frac{\hbar^2}{2m_2} \nabla^2_{r_2} + V(r_1, r_2) \right) \Psi_a(r_1, r_2)$$

The left side of this equation depends of $t$ independent of $r_1$ and $r_2$ while the right side depends on $r_1$ and $r_2$ independent of $t$.

This can hold for all $r_1$, $r_2$ and $t$ if both sides of this equation are equal to the same constant $E$. In this case we have 2 equations:

$$i\hbar \frac{\partial \Psi_b(t)}{\partial t} = E\Psi_b(t)$$

$$\left( -\frac{\hbar^2}{2m_1} \nabla^2_{r_1} - \frac{\hbar^2}{2m_2} \nabla^2_{r_2} + V(r_1, r_2) \right) \Psi_a(r_1, r_2) = E\Psi_a(r_1, r_2)$$
The second equation above is called the **time-independent Schrödinger equation**. The solution of the first equation is

\[ \Psi_b(t) = \Psi_b(0)e^{-i\frac{Et}{\hbar}} 0 \]

The full solution is

\[ \Psi(r_1, r_2, t) = \Psi_a(r_1, r_2)e^{-i\frac{Et}{\hbar}} \] (1)

where the constant factor \( \Psi_b(0) \) has been absorbed in \( \Psi_a(r_1, r_2) \).

Note that (1) is not the only solution. The Schrödinger equation is a linear equation so if \( \Psi_A(r_1, r_2, t) \) and \( \Psi_B(r_1, r_2, t) \) are both solutions to

\[ i\hbar \frac{\partial\Psi(r_1, r_2, t)}{\partial t} = \left( -\frac{\hbar^2}{2m_1} \nabla^2_1 - \frac{\hbar^2}{2m_2} \nabla^2_2 + V(r_1, r_2) \right) \Psi(r_1, r_2, t) \]

then any linear combination of these solutions also satisfies the same Schrödinger equation

\[ \Psi(r_1, r_2, t) = \alpha \Psi_A(r_1, r_2, t) + \beta \Psi_B(r_1, r_2, t) \]

are also solutions, where \( \alpha \) and \( \beta \) are complex constant.

The freedom to use arbitrary linear combinations of the solution is needed to satisfy the initial conditions of the problem.

**Additional simplifications - non-interacting particles**

In this case

\[ V(r_1, r_2, t) = V_1(r_1) + V_2(r_2) \]

This is still a time independent potential so we can look for a solution of the form

\[ \Psi(r_1, r_2, t) = \Psi_a(r_1, r_2)e^{-i\frac{Et}{\hbar}} \]

where

\[ \left( -\frac{\hbar^2}{2m_1} \nabla^2_1 - \frac{\hbar^2}{2m_2} \nabla^2_2 + V_1(r_1) + V_2(r_2) \right) \Psi_a(r_1, r_2, t) = E\Psi_a(r_1, r_2) \]

In order to solve this equation look for solutions of the form

\[ \Psi_a(r_1, r_2) = \Psi_1(r_1)\Psi_2(r_2) \]

In this case the time-independent Schrödinger equation becomes
\[
\frac{1}{\Psi_1(r_1)} \left( -\frac{\hbar^2}{2m_1} \nabla^2_{r_1} + V_1(r_1) \right) \Psi_1(r_1) = \\
E - \frac{1}{\Psi_2(r_2)} \left( -\frac{\hbar^2}{2m_1} \nabla^2_{r_2} + V_2(r_2) \right) \Psi_2(r_2)
\]

where both sides must be equal to the same constant \( E_1 \). This gives 2 uncoupled equations for \( \Psi_1(r_1) \) and \( \Psi_1(r_1) \):

\[
\left( -\frac{\hbar^2}{2m_1} \nabla^2_{r_1} + V_1(r_1) \right) \Psi_1(r_1) = E_1 \Psi_1(r_1)
\]

\[
\left( -\frac{\hbar^2}{2m_1} \nabla^2_{r_2} + V_2(r_2) \right) \Psi_2(r_2) = E_2 \Psi_2(r_2)
\]

where \( E = E_1 + E_2 \). In this case the two particle solutions are products of two one-particle solutions:

\[
\Psi(r_1, r_2, t) = \Psi_1(r_1) e^{-i \frac{E_1}{\hbar} t} \Psi_2(r_2) e^{-i \frac{E_2}{\hbar} t}
\]

For a wave functions that are products single particle states, the state of particle 1 depends on the state of particle 2. In other words measuring states of particle 1 determines the state of particle 2. States that are products of single-particle states are called pure states.

Linear combinations of product states

\[
\Psi(r_1, r_2, t) = \alpha \Psi_{1a}(r_1) e^{-i \frac{E_{1a}}{\hbar} t} \Psi_{2a}(r_2) e^{-i \frac{E_{2a}}{\hbar} t} \\
+ \beta \Psi_{1b}(r_1) e^{-i \frac{E_{1b}}{\hbar} t} \Psi_{2b}(r_2) e^{-i \frac{E_{2b}}{\hbar} t}
\]

where \( E = E_{1a} + E_{2a} = E_{1b} + E_{2b} \) are also solutions of the Schrödinger equation with the same total energy. In this case, when the states cannot be written as products of solutions, they are called entangled states. The term entangled arises because measuring the state of one of the particles does not fix the state of the other particle.

**Additional simplifications - local potentials**

Local potentials have the form

\[
V(r_1, r_2, t) = V(r_1 - r_2)
\]
This is still a time-independent potential. This can be simplified by changing coordinates from single-particle coordinates to center of mass and relative coordinates.

\[ \mathbf{R} := \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \]

\[ \mathbf{r} := \mathbf{r}_1 - \mathbf{r}_2 \]

In this case the chain rule gives

\[ \nabla \mathbf{r}_1 = \frac{m_1}{m_1 + m_2} \nabla_{\mathbf{R}} + \nabla \mathbf{r} \]

\[ \nabla \mathbf{r}_2 = \frac{m_2}{m_1 + m_2} \nabla_{\mathbf{R}} - \nabla \mathbf{r} \]

Squaring gives

\[ \nabla^2 \mathbf{r}_1 = \left( \frac{m_1}{m_1 + m_2} \right)^2 \nabla^2_{\mathbf{R}} + \nabla^2 \mathbf{r} + 2 \frac{m_1}{m_1 + m_2} \nabla_{\mathbf{R}} \cdot \nabla \mathbf{r} \]

and

\[ \nabla^2 \mathbf{r}_2 = \left( \frac{m_2}{m_1 + m_2} \right)^2 \nabla^2_{\mathbf{R}} + \nabla^2 \mathbf{r} - 2 \frac{m_2}{m_1 + m_2} \nabla_{\mathbf{R}} \cdot \nabla \mathbf{r} \]

If we divide the first equation by \( m_1 \) and the second by \( m_2 \) and add the cross terms involving both variable cancel. What remains is

\[ -\frac{\hbar^2}{2m_1} \nabla^2 \mathbf{r}_1 - \frac{\hbar^2}{2m_2} \nabla^2 \mathbf{r}_2 = \]

\[ -\frac{\hbar^2}{2}\left( \frac{1}{m_1 + m_2} \right) \nabla^2_{\mathbf{R}} + \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \nabla^2 \mathbf{r} \]

\( M = m_1 + m_2 \) is called the total mass of this system while \( \mu \) defined by \( \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \) is called the reduced mass. In terms of these new variables the Hamiltonian becomes

\[ H = -\frac{\hbar^2}{2M} \nabla^2_{\mathbf{R}} - \frac{\hbar^2}{2\mu} \nabla^2 \mathbf{r} + V(\mathbf{r}) \]

In this case the wave function can be written in the form

\[ \Psi(\mathbf{R}, \mathbf{r}, t) = e^{-i(E_{\mathbf{R}} + E_{\mathbf{r}})t} \Psi_{\mathbf{R}}(\mathbf{r}) \Psi_{\mathbf{r}}(\mathbf{r}) \]

where

\[ -\frac{\hbar^2}{2M} \nabla^2_{\mathbf{R}} \Psi_{\mathbf{R}}(\mathbf{R}) = E_{\mathbf{R}} \Psi_{\mathbf{R}}(\mathbf{R}) \]
\[
\left(-\frac{\hbar^2}{2\mu} \nabla^2_x + V(r)\right) \Psi_r(r) = E_r \Psi_r(r)
\]

The solution of the first equation has the form

\[
\Psi_R(R) = e^{ikR}
\]

were

\[
E_R = \frac{\hbar^2 k \cdot k}{2M}
\]

and \(k\) is a constant vector. The quantity \(E_R\) is the center of mass energy of this state. This case \(P := \hbar k\) is the center of mass momentum.

In this case the solution has the form

\[
\Psi(R, r, t) = e^{ikR} \Psi_r(r) e^{-i(E_r + \frac{P \cdot P}{2M})} \frac{1}{\hbar}
\]

Remark - I have used a sloppy notation. Actually \(\Psi(R, r, t)\) and \(\Psi(r_1, r_2, t)\) are different functions. If I call them \(\Psi_{Rr}(R, r, t)\) and \(\Psi_{12}(r_1, r_2, t)\) they are related by

\[
\Psi_{12}(r_1, r_2, t) = \Psi_{Rr}(R(r_1, r_2), r(r_1, r_2), t) \sqrt{\left| \frac{\partial(R, r)}{\partial(r_1, r_2)} \right|}
\]

The last factor is a Jacobian that maintains the correct normalization when the variables are changed. It satisfies

\[
d^3Rd^3r = \left| \frac{\partial(R, r)}{\partial(r_1, r_2)} \right| d^3r_1d^3r_2
\]

so

\[
\int \Psi_{12}^*(r_1, r_2, t)\Psi_{12}(r_1, r_2, t) d^3r_1d^3r_2 = \int \Psi_{Rr}^*(R(r_1, r_2), r(r_1, r_2), t)\Psi_{Rr}(R(r_1, r_2), r(r_1, r_2), t) \left| \frac{\partial(R, r)}{\partial(r_1, r_2)} \right| d^3r_1d^3r_2 = \int \Psi_{Rr}^*(R, r, t)\Psi_{Rr}(R, r, t) d^3Rd^3r
\]

In this case the Jacobian is the product of the determinant of the matrices:

\[
\frac{\partial(R, r)}{\partial((r_1, r_2))} = \det \left( \begin{array}{cc} \frac{\partial X}{\partial x_1} & \frac{\partial X}{\partial x_2} \\ \frac{\partial Y}{\partial y_1} & \frac{\partial Y}{\partial y_2} \end{array} \right) \det \left( \begin{array}{cc} \frac{\partial Y}{\partial z_1} & \frac{\partial Y}{\partial z_2} \\ \frac{\partial Z}{\partial x_1} & \frac{\partial Z}{\partial x_2} \end{array} \right) \det \left( \begin{array}{cc} \frac{\partial Z}{\partial y_1} & \frac{\partial Z}{\partial y_2} \\ \frac{\partial X}{\partial z_1} & \frac{\partial X}{\partial z_2} \end{array} \right) = \]
\[
\left( \det \left( \frac{m_1}{m_1 + m_2} \quad \frac{m_2}{m_1 + m_2} \right) \right)^3 = -1
\]

In this case all of these factors are 1.

Remark - in general I will continue using this sloppy notation. The interpretation should be clear from context.

**Identical particles**

If both particles are identical and \( V(r_1, r_2) = V(r_2, r_1) \) it follows that

\[
H(r_1, r_2, t) := \left( -\frac{\hbar^2}{2m} \nabla^2_{r_1} - \frac{\hbar^2}{2m} \nabla^2_{r_2} + V(r_1, r_2, t) \right) = \nabla^2_{r_2} - \frac{\hbar^2}{2m} \nabla^2_{r_1} + V(r_2, r_1, t) = H(r_2, r_1, t)
\]

This means that if we interchange particle 1 and particle 2 the Hamiltonian does not change. It we define the particle exchange operator \( P_{12} \) by

\[
P_{12} \Psi(r_1, r_2, t) = \Psi(r_2, r_1, t)
\]

It follows that

\[
P_{12} H(r_1, r_2, t) \Psi(r_1, r_2, t) = H(r_2, r_1, t) \Psi(r_2, r_1, t) = H(r_1, r_2, t) P_{12} \Psi(r_1, r_2, t)
\]

or abstractly

\[
P_{12} H = H P_{12}
\]

What this means is that if

\[
H \Psi = E \Psi
\]

and this is multiplied by \( P_{12} \) then

\[
P_{12} H \Psi = E P_{12} \Psi = H P_{12} \Psi
\]

This requires that both

\[
\Psi P_{12} \quad \text{and} \quad \Psi
\]

are both eigenstates of \( H \) with the same eigenvalue.

Since two applications of \( P_{12} \) leave the particles unchanged,

\[
P_{12}^2 = 1
\]
If $\Psi$ is an eigenstate of $H$ with energy $E$ then

$$\Psi_s := N_s(\Psi + P_{12}\Psi) \quad \Psi_a := N_a(\Psi - P_{12}\Psi)$$

are simultaneous eigenstates of $H$ with energy $E$ and $P_{12}$ with eigen-value $\pm$. Here $N_a$ and $N_b$ are normalization coefficients.

There is result from relativistic quantum field theory, called the **spin statistics theorem**, asserts that the wave functions for particles with integer spin must symmetric with respect to interchanging particles while identical particles with half integer spins must be in anti-symmetric with respect to interchanging particles.

While the proof of the spin statistic theorem is beyond the scope of this class the basic concepts are not hard to follow. Fields are things like temperature that have values at different points and times. The temperatures at 2 different points and time can impact each other only if we wait long enough for heat to flow form one point to the other. For quantum fields at different points they can only impact each other if it is possible to send a light signal from one to the other. In addition to scalar fields like the temperature, there are vector fields (like wind velocity) and spinor fields in quantum field theory. For fields with angular momentum these parts involve even or odd degree polynomials in the momentum of the field. In order to ensure that signals cannot violate causality on very small distances it turns out the fields with odd degree polynomials (half integer spin) must anticommute at space-like separation while fields with even degree polynomials (integer spin) must commute at space-like separations.

Particles with integer spins are called **Bosons** while particles with half integer spin are called **Fermions**.

The wave functions for particle’s with spins have the form

$$\Psi(r, \mu, t)$$

with the interpretation that

$$dP = \Psi^*(r, \mu, t)\Psi(r, \mu, t)d^3r$$

is the differential probability for finding a particle with spin projection $\mu$ within $d^3r$ of $r$. In this case the normalization condition

$$\int \sum_{\mu=-j}^j \Psi^*(r, \mu, t)\Psi(r, \mu, t)d^3r = 1$$
includes a sum over spins.

For states of two identical Bosons the wave functions are symmetric on interchange of particles 1 and 2:

$$\Psi_{12}(r_1, \mu_1, r_2, \mu_t, t) = N_s (\Psi_1(r_1, \mu_1, t)\Psi_2(r_2, \mu_2, t) + \Psi_2(r_1, \mu_1, t)\Psi_1(r_2, \mu_2, t))$$

while for two identical Fermions the wave functions are antisymmetric on interchange of particles 1 and 2

$$\Psi_{12}(r_1, \mu_1, r_2, \mu_t, t) = N_a (\Psi_1(r_1, \mu_1, t)\Psi_2(r_2, \mu_2, t) - \Psi_2(r_1, \mu_1, t)\Psi_1(r_2, \mu_2, t))$$

where $N_s$ and $N_a$ are constant coefficients chosen to ensure that the probability is 1.

These states have the property that

$$P_{12} \Psi_{12a} = -\Psi_{12a} \quad P_{12} \Psi_{1bs} = \Psi_{12s}$$

Since

$$i\hbar \frac{\partial}{\partial t} (I \mp P_{12}) \Psi_{12} = (I \mp P_{12})H \Psi_{12} = H(I \mp P_{12}) \Psi_{12}$$

if

$$(I \mp P_{12}) \Psi_{12} = 0$$

it remains so for all time.

When the single particle states are orthogonal

$$\int \sum_{\mu=-j}^j \Psi_1^*(r, \mu, t)\Psi_2^*(r, \mu, t)dr = 0$$

then the normalization constants are $N_a = N_s = \frac{1}{\sqrt{2}}$:

$$\Psi_{12a}(r_1, \mu_1, r_2, \mu_t, t) = \frac{1}{\sqrt{2}} (\Psi_1(r_1, \mu_1, t)\Psi_2(r_2, \mu_2, t) - \Psi_2(r_1, \mu_1, t)\Psi_1(r_2, \mu_2, t))$$

for identical Bosons

$$\Psi_{12a}(r_1, \mu_1, r_2, \mu_t, t) = \frac{1}{\sqrt{2}} (\Psi_1(r_1, \mu_1, t)\Psi_2(r_2, \mu_2, t) - \Psi_2(r_1, \mu_1, t)\Psi_1(r_2, \mu_2, t))$$

for identical Fermions

For a given pair of orthogonal single-particle wave functions consider the expectation value of the square of the distance between the particle
\[ \langle (r_1 - r_2)^2 \rangle = \]
\[ \int \Psi_{12x}^*(r_1, \mu_1, r_2, \mu_2, t)(r_1 - r_2)^2 \Psi_{12x}(r_1, \mu_1, r_2, \mu_2, t) = \]
\[ \frac{1}{2} \int d^3r_1 d^3r_2 \sum_{\mu_1 \mu_2} (\Psi_a^*(r_1, \mu_1, t) \Psi_b^*(r_2, \mu_2, t) \pm \Psi_b^*(r_1, \mu_1, t) \Psi_a^*(r_2, \mu_2, t)) \times \]
\[ (r_1 \cdot r_1 + r_2 \cdot r_2 - 2r_1 \cdot r_2) (\Psi_a(r_1, \mu_1, t) \Psi_b(r_2, \mu_2, t) \pm \Psi_b(r_1, \mu_1, t) \Psi_a(r_2, \mu_2, t)) \]
This expression has 12 different integrals.
\[ \frac{1}{2} \int \Psi_a^*(r_1, \mu_1, t) \Psi_a(r_1, \mu_1, t) \Psi_b^*(r_2, \mu_2, t) \Psi_b(r_2, \mu_2, t)(r_1 \cdot r_1 + (r_1 \cdot r_2) - 2r_1 \cdot r_2) dr_1 dr_2 + \]
\[ \frac{1}{2} \int \Psi_b^*(r_1, \mu_1, t) \Psi_b(r_1, \mu_1, t) \Psi_a^*(r_2, \mu_2, t) \Psi_a(r_2, \mu_2, t)(r_1 \cdot r_1 + r_2 \cdot r_2 - 2r_1 \cdot r_2) dr_1 dr_2 + \]
\[ \pm \frac{1}{2} \int \Psi_b^*(r_2, \mu_2, t) \Psi_b(r_2, \mu_2, t) \Psi_a^*(r_1, \mu_1, t) \Psi_a(r_1, \mu_1, t)(r_1 \cdot r_1 + r_2 \cdot r_2 - 2r_1 \cdot r_2) dr_1 dr_2 \]
\[ \pm \frac{1}{2} \int \Psi_a^*(r_1, \mu_1, t) \Psi_b(r_1, \mu_1, t) \Psi_b^*(r_2, \mu_2, t) \Psi_a(r_2, \mu_2, t)(r_1 \cdot r_1 + r_2 \cdot r_2 - 2r_1 \cdot r_2) dr_1 dr_2 = \]
The first of these 4 terms is
\[ \frac{1}{2} (\langle r \cdot r \rangle_a + \langle r \cdot r \rangle_b - 2 \langle r \rangle_a \cdot \langle r \rangle_b) \]
The second one is
\[ \frac{1}{2} (\langle r \cdot r \rangle_b + \langle r \cdot r \rangle_a - 2 \langle r \rangle_b \cdot \langle r \rangle_a) \]
In the third and fourth term the orthogonality of the different wave functions is used. The terms that survives are the mixed terms
\[ \pm \frac{1}{2} (-2a \langle r \rangle_b \cdot \langle r \rangle_a - 2b \langle r \rangle_a \cdot \langle r \rangle_b) \]
Putting everything together gives
\[ \langle r \cdot r \rangle_a + \langle r \cdot r \rangle_b - 2\langle r \rangle_b \cdot \langle r \rangle_a \pm 2b \langle r \rangle_a \cdot \langle r \rangle_b \]
The first two terms are identical for Bosons or Fermions. The term with both signs is positive
\[ 2b \langle r \rangle_a \cdot \langle r \rangle_b = 2|b\langle r \rangle_a|^2 \]
It is negative for the upper sign (Bosons) and positive for the lower sign (Fermions).

This shows independent of any forces, the mean distance between identical Fermions is larger than the mean distance between identical fermions. Note the term

\[ a \langle r \rangle_b = \int \Psi^*_a(r, \mu_1, t) r \Psi_b(r, \mu_1, t) d^3r \]

vanishes if

\[ \Psi^*_a(r, \mu_1, t) \Psi_b(r, \mu_1, t) = 0 \]

for all \( r \). This means that when the identical particles are far apart, they behave like distinguishable particles. The identity becomes important when the particles are close.

For spin states you learned last semester that two spin 1/2 particles could be in a symmetric (spin 1) state or an antisymmetric (spin 0) state. A two particle wave function could also be made out of products of spin states of a given symmetry and spatial states of a given symmetry.

If the particles are fermions the overall wave functions must be antisymmetric. For two spin 1/s fermions, if the spin is 1 (symmetric) the spatial wave function must be antisymmetric, while if the spin is 0 (antisymmetric) the spatial part of the wave functions must be symmetric.

So far we have only talked about 2 particle systems. For systems of many identical particles the entire wave function must be antisymmetric for Fermions and symmetric for Bosons.

An interesting case is a system of particles confined to a box of side length \( a \). The time independent Schrödinger equation for each particle is

\[-\frac{\hbar^2}{2m} \nabla^2 \Psi(r) = E \Psi(r)\]

This can be solved by separation of variables. The single particle wave functions are products of the form

\[ \Psi(r) = \Psi_x(x) \Psi_y(y) \Psi_z(z) \]

The individual wave functions in \( x, y \) and \( z \) are

\[ \Psi_x(x) = A_x \sin(kx) + B_x \cos(kx) \]
where
\[ E = E_x + E_y + E_z \]
and
\[ E_x = \frac{\hbar^2 k_x^2}{2m} \quad E_y = \frac{\hbar^2 k_y^2}{2m} \quad E_z = \frac{\hbar^2 k_z^2}{2m} \]
The boundary conditions require that these functions vanish a 0 and \( a \). This will be satisfied if
\[ k_x = \frac{n_x \pi}{a} \quad \Psi_x(x) = A_x \sin\left(\frac{n_x \pi x}{a}\right) \]
The normalization requires
\[ 1 = \int_0^a A^2 \sin^2\left(\frac{n \pi x}{a}\right) dx = \int_0^{n \pi} A^2 \sin^2(u) \frac{a}{n \pi} du = A^2 \frac{n \pi}{2} \cdot \frac{a}{n \pi} = A^2 \frac{a}{2} \]
where \( u = n \pi x / a \) which gives
\[ A = \sqrt{\frac{2}{a}} \]
The full wave function becomes
\[ \Psi(r) = \left(\frac{2}{a}\right)^{3/2} \sin\left(\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_y \pi y}{a}\right) \sin\left(\frac{n_z \pi z}{a}\right) \]
states of a single particle are labeled by three positive integers \((n_x, n_y, n_z)\) with energy
\[ E = \frac{\hbar^2 \pi^2}{2ma^2} (n_x^2 + n_y^2 + n_z^2) \]
For spinless particles the ground state of the system has every particle in the \((1,1,1)\) state with total energy
\[ E_{\text{min}} = N^3 \frac{3\hbar^2 \pi^2}{2ma^2} \]
On the other hand if these were spin 1/2 particles, the antisymmetrization requires that the wave function vanishes unless every particle is in different state. Since each particle has 2 spin states, only two particles can be in the \((1,1,1)\) state. There are 2 particles in each of the \((1,1,2)\), \((1,2,1)\), and \((2,1,1)\) states. These have higher energy than the ground state. Each time that a new particle is added it must be put in an unoccupied state.
We can try to count states. We can think of a three dimensional grid where the grid point are separated by integers. In this case each grid point represents 1 state. We can count states by starting at one fixed corner and labeling state by counting the number of shifts by integers. The total number of states will be approximately equal the volume of the cube. If we replace the cube by a sphere

\[ N = \frac{4\pi}{3} n^3 \]

where \( n \) is the radius for the sphere. However in this problem all of the \( n \) values must be positive so the number of states of a given spin is actually the volume of the positive part of the sphere.

\[ N = \frac{1}{6} \left( \frac{4\pi}{3} n^3 \right) = \frac{\pi}{3} n^3 \]

Since the particles have spin 1/2 the actual number of particles is twice this:

\[ N = 2 \frac{1}{6} \left( \frac{4\pi}{3} n^3 \right) = \frac{\pi}{3} n^3 \]

This \( n \) is associated with the energy of the highest energy particle. The gives

\[ n^2 = \frac{3N^{2/3}}{\pi} \]

For \( n^2 = n_x^2 + n_y^2 + n_z^2 \) the energy is

\[ E = \frac{\pi^2 \hbar^2 n^2}{2ma^2} \]

and the energy of the highest energy state is

\[ E = \frac{\hbar^2 \pi^2}{2ma^2} n^2 = \frac{\hbar^2 \pi^2}{2ma^2} \left( \frac{3N\pi}{\pi} \right)^{2/3} = \frac{\hbar^2}{2m} \left( \frac{3N\pi^3}{a^3\pi} \right)^{2/3} \]

This gives

\[ E = \frac{\hbar^2}{2m} \left( \frac{3N\pi^2}{a^3} \right)^{2/3} \]

we see that the energy of the last added Fermion increases by the \( 2/3 \) power of the number of particles (states). This energy is called the Fermi energy. In this case the “Fermi repulsion” keeps the system falling into the ground state. Note that \( N/a^3 = \rho \) is the number of particles per unit volume.