1. Consider an infinitely deep square well of width $a$. Assume that the edges of the well are at $x = 0$ and $x = a$.

   a. What are the unit normalized eigenfunctions for a particle of mass $m$ in the well?
   b. What are the energy eigenvalues for the particle?
   c. What are the unit normalized eigenfunctions of a state of two identical mass $m$ spin 0 particles in this well.
   d. What are the unit normalized eigenfunctions of a state of two identical mass $m$ spin 1/2 particles in this well.
   e. What is the ground state energy if there are three spin 1/2 particles in the well.
   f. What is the ground state energy if there are three spin 0 particles in the well.

2. Vector operators $\vec{V}$ satisfy $[L^i, V^j] = i\hbar \sum_{k=1}^3 \epsilon_{ijk} V^k$. Assume that $H$ commutes with $\vec{L}$.

   a. Assume that $[H, V^1] = 0$. Show that $[H, \vec{V}] = 0$.
   b. Using the result from part a show $[H, \vec{V} \cdot \vec{V}] = 0$
   c. Show that $\vec{V} \cdot \vec{V}$ is a conserved quantity.
   d. Assume that you know $\langle n, l, m|V^x|n', l', m'\rangle$, $\langle n, l, m|L^x|n', l', m'\rangle$, $\langle n, l, m|L^z|n', l', m'\rangle$. Find $\langle n, l, m|V^z|n', l', m'\rangle$.

3. Consider an unperturbed Hamiltonian of the form

   $$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \quad E_1 \neq E_2$$

   and a small perturbation of the form

   $$V = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$$

   a. Calculate the unperturbed eigenvectors and eigenvalues
   b. Calculate the first order correction to the eigenvalues
   c. Calculate the second order correction to the eigenvalues
   d. Calculate the exact eigenvalues of the full Hamiltonian.

4. Consider the Hamiltonian of problem 3 but with $E_1 = E_2 = E_0$. 
a. Use degenerate perturbation theory to find approximate eigenvalues of $H_0 + V$

b. Find the corresponding eigenvectors.

c. Is your solution exact or approximate.

d. What is the ground state energy of this system (assume $\lambda > 0$)?
Solutions

1a.  
\[ \langle x | n \rangle = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \quad n = 1, 2, 3, \ldots \]

1b.  
\[ E_n = \frac{\hbar^2 \pi^2 n^2}{2ma^2} \]

c.  
\[ \langle x, y | n, m \rangle = \sqrt{\frac{2}{a}} \left( \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) + \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \right) \]

d.  
\[ \langle x, y | n, m \rangle = \sqrt{\frac{2}{a}} \left( \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) - \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \right) \]

e.  
\[ E = \frac{\hbar^2 \pi^2}{2ma^2} \left(1^2 + 1^2 + 2^2\right) = \frac{6\hbar^2 \pi^2}{2ma^2} \]

1f.  
\[ E = 3\frac{\hbar^2 \pi^2}{2ma^2} \]

2a.  
\[ [H, V^3] = \frac{i}{\hbar} [H, [L^2, V^1]] = \frac{i}{\hbar} ([H, L^2], V^1) + i[L^2, [H, V^1]] = 0 \]
\[ [H, V^2] = -\frac{i}{\hbar} [H, [L^3, V^1]] = -\frac{i}{\hbar} ([H, L^3], V^1) + i[L^3, [H, V^1]] = 0 \]

these vanish since \([H, V^1] = [H, L^1] = [H, V^2] = [H, L^2] = 0\).

2b.  
\[ [H, \vec{V} \cdot \vec{V}] = \vec{V} \cdot [H, \vec{V}] + [H, \vec{V}] \cdot \vec{V} = 0 \]

2c.  
\[ \frac{d\vec{V} \cdot \vec{V}}{dt} = \frac{d}{dt} e^{iHt/\hbar} \vec{V} e^{-iHt/\hbar} = \frac{i}{\hbar} e^{iHt/\hbar} [H, \vec{V} \cdot \vec{V}] e^{-iHt/\hbar} = 0 \]

2d.  
By the Wigner Eckart theorem

\[ \frac{\langle n, l, m | V^2 | n', l', m' \rangle}{\langle n, l, m | V^2 | n', l', m' \rangle} = \frac{\langle n, l, m | L^2 | n', l', m' \rangle}{\langle n, l, m | L^2 | n', l', m' \rangle} \]

This means

\[ \langle n, l, m | V^2 | n', l', m' \rangle = \langle n, l, m | L^2 | n', l', m' \rangle \frac{\langle n, l, m | L^2 | n', l', m' \rangle}{\langle n, l, m | L^2 | n', l', m' \rangle} \]
3a. 

\[ E_1 \psi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad E_2 \psi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

3b. The first order correction is zero because the perturbation has no diagonal elements in the basis of eigenstates.

3c. The second order corrections are

\[ \Delta E_1 = \frac{|\langle 1 | V | 2 \rangle|^2}{E_1 - E_2} = \frac{\lambda^2}{E_1 - E_2} \]

\[ \Delta E_2 = \frac{|\langle 2 | V | 1 \rangle|^2}{E_2 - E_1} = \frac{\lambda^2}{E_2 - E_1} \]

3d. we need to find the zeroes of

\[ (E - E_1)(E - E_2) - \lambda^2 = 0 = E^2 - (E_1 + E_2)E - \lambda^2 \]

they are

\[ E = \frac{(E_1 + E_2)}{2} \pm \frac{1}{2} \sqrt{(E_1 + E_2)^2 + 4\lambda^2} = \frac{(E_1 + E_2)}{2} (1 \pm \sqrt{1 + \frac{\lambda^2}{(E_1 + E_2)^2}}) \]

4. Consider the Hamiltonian of problem 3 but with \( E_1 = E_2 = E_0 \).

4a. In this case the matrix on the degenerate subspace is

\[ \begin{pmatrix} E - E_0 & -\lambda \\ -\lambda & E - E_0 \end{pmatrix} \]

It has eigenvectors

\[ E_{\pm} = E_0 \pm \lambda \]

4b. The eigenvectors are solutions of

\[ \begin{pmatrix} \mp \lambda & -\lambda \\ -\lambda & \pm \lambda \end{pmatrix} \begin{pmatrix} a_+ \\ b_+ \end{pmatrix} = 0 \]

\[ \psi_{\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \mp 1 \end{pmatrix} \]

4c. The solutions are exact since the above matrix is the full Hamiltonian matrix.

4d. \( E_- = E_0 - \lambda = \) ground state energy.