Multiple valued functions

**Example** \( z = \ln Z \)

\[ e^{\ln z} = z \rightarrow \ln z = \ln |z| + i(\phi + 2\pi n) \]

As \( \phi \) increases from \( 0 \rightarrow 2\pi \), the value of \( \ln z \) changes by \( 2\pi i \).

**Definition** \( z_0 \) is a branch point of \( f(z) \) if for every closed curve around \( z_0 \), \( f(z) \) does not return to its original value.

\( z \) is a branch point of \( \ln z \)

Since \( z = 1z_1 e^{i\phi} = 1 \rightarrow f(z) \neq f(1z_1 e^{2\pi n i}) \)

\( z \) is a branch point since

\[ \ln z = -\ln \frac{1}{z} \quad (z \rightarrow \frac{1}{z} \rightarrow \infty) \]

\( |z| = \infty \) is also a branch point.

**Definition** a branch cut is a continuous curve connecting 2 branch points.
branch points are properties of \( f(z) \); there is considerable freedom in choosing branch cuts.

**Example 1**

\[
\ln z = \ln |z| + i(\phi + 2\pi n_i)
\]

Choose the cut to go from 0 to \( a \) along the positive real axis.

The exposed part of the plane \( 2\pi n < \phi < 2(\pi + 1)\pi \) is called the \( n^{th} \) Riemann sheet of the Riemann surface.

**Example 2**

\[
f(z) = z^{1/2}
\]

For \( z = |z| e^{i\phi} \)

\[
f(z) = |z|^{1/2} e^{i\phi/2}
\]

As \( \phi \) increases from 0 to \( 2\pi \), \( f(z) \) changes sign. This means \( z = 0 \) is a branch point.
Also note that $g(\frac{1}{2}) = \frac{i}{\sqrt{2}} = \frac{\sqrt{2}}{2} \frac{-i\pi}{2}$.

If we let $z = |z|e^{i\phi}$, then $f(\frac{1}{2}) \rightarrow -f(\frac{1}{2})$ as $\phi$ increases from $0 \rightarrow 2\pi$.

This means $z = \infty$ is also a branch point.

Possible branch cut is $\sqrt{z}$ as $\phi : 0 \rightarrow 4\pi$.

$$\sqrt{ze^{4\pi i}} = \sqrt{z}$$

$$\sqrt{\frac{1}{z}e^{-4\pi i}} = \sqrt{\frac{1}{z}}$$

This function returns to its original value as $\phi : 0 \rightarrow 4\pi$. The Riemann surface has 2 Riemann sheets.

$\frac{1}{n}$

has $n$ sheets with branch points at $0$, $\infty$. 
Consider
\[ \frac{1}{\sqrt{z^2-1}} = \frac{1}{\sqrt{z-1}} \frac{1}{\sqrt{z+1}} \]

This function has branch points \( c = \pm 1 \).

At \( z = 1 \), \( z = -1 \)

but if we let \( z = \frac{1}{2} \)

\[ \frac{1}{\sqrt{\frac{1}{2}-1}} = \frac{2}{\sqrt{1-z^2}} = \frac{2}{\sqrt{(1-2)(1+2)}} \]

which has branch points \( a = \pm 1 \),

but not at \( \infty \). In this case the two branch points

"at \( \infty \)" cancel

It is possible to perform
continuum integrals on multiple
valued functions.

- Pick one of the Riemann sheets
- Draw a curve on that sheet
  that does not cross the
  branch cut
- If the function is meromorphic
  in the cut plane we can use
  Cauchy's theorem + the
  residue theorem on the Riemann sheet
Example

Consider the integral

\[ \int_{0}^{\infty} \frac{x^{p-1}}{x^{2}+1} \, dx \quad 0 < p < 2 \]

If \( p = 0 \), this behaves like \( \int \frac{dx}{x} = \ln x \) near \( x = 0 \), which is infinite, while if \( p = 2 \), this behaves like \( \int \frac{dx}{x} = \ln x \) near \( x = \infty \). For \( 0 < p < 2 \), the integral converges.

If \( p \neq 1 \), the integrand is a multi-valued function.

To treat this, choose the Riemann sheet consisting of the points

\[ z = \left| z \right| e^{i\phi} \quad 0 < \phi < 2\pi \]

Choose the cut along the positive real axis.

To evaluate this integral, consider the closed contour \( \gamma \) on this Riemann sheet.
This curve is in the cut plane. In the cut plane the curve encloses 2 poles of \( \mathcal{S}(z) \) at \( z = e^{\frac{i\pi}{3}}, e^{\frac{2i\pi}{3}} \).

The curve consists of 4 parts:

1. \( Z = x^2 \int_0^\infty \mathcal{S}(x) \, dx \)

2. \( Z = R e^{i\phi} \int_0^{2\pi} \mathcal{S}(R e^{i\phi}) \, i R e^{i\phi} \, d\phi \)

3. \( Z = x e^{2ni} \int_0^\infty \mathcal{S}(x e^{2ni}) \, dx \)

4. \( Z = r e^{i\phi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathcal{S}(r e^{i\phi}) \, i r e^{i\phi} \, d\phi \)

We are interested in the limit \( R \to \infty, r \to 0 \):

1. \( \int_0^\infty \frac{x^{p-1}}{x^2 + 1} \, dx \)

2. \( \int_0^{2\pi} i R e^{i\phi} \left( \frac{R e^{i\phi}}{(R^2 e^{2i\phi} + 1)} \right)^{p-1} \sim \frac{R^p}{R^2} \to 0 \) as \( R \to \infty \)

3. \( \int_0^\infty \frac{(x e^{2ni})^{p-1}}{(x^2 e^{2ni} + 1)} \, dx e^{2ni} \)

4. \( \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} i R e^{i\phi} \left( \frac{R e^{i\phi}}{R^2 e^{2i\phi} + 1} \right)^{p-1} \to \frac{R^p}{R^2} \to 0 \) as \( R \to \infty \)
This shows that in the limit \( r \to 0 \) \( R \to \infty \) the only contribution to the contour integral comes from regions I and III.

The contour encloses two poles — one at \( Z_0 = e^{i\pi} \) and one at \( Z_0 = e^{3i\pi/2} \).

Note we cannot use \( e^{i\pi} \) in our chosen cut plane because going from \( 0 \to e^{i\pi} \) would cross the branch cut.

Using the residue theorem gives

\[
2\pi i \left( \frac{e^{i\pi/2}(p-1)}{e^{i\pi/2} - e^{3i\pi/2}} + \frac{e^{3i\pi/2}(p-1)}{e^{3i\pi/2} - e^{i\pi/2}} \right)
\]

\[
\frac{2\pi i}{e^{i\pi}} (i\pi(p-1)) \left( \frac{e^{-i\pi/2}(p-1)}{e^{-i\pi/2} - e^{i\pi}} + \frac{e^{i\pi/2}(p-1)}{e^{i\pi/2} - e^{-i\pi}} \right)
\]

Multiply numerator and denominator by \( \frac{2i}{2i} \)

\[
\frac{2\pi i}{e^{i\pi}} \frac{i\pi(p-1)}{(e^{i\pi/2} - e^{-i\pi/2})/2i} = \frac{2\pi i}{e^{i\pi}} \frac{i\pi(p-1)}{(e^{i\pi/2} - e^{-i\pi/2})/2i}
\]

\[
\frac{2\pi i}{e^{i\pi}} \frac{i\pi(p-1)}{\sin \left( \frac{\pi}{2}(p-1) \right)} = \frac{\sin \left( \frac{\pi}{2}(p-1) \right)}{\sin \left( \frac{\pi}{2} \right)}
\]
\( m_{11} \) is equal to the sum of the integrals \( I \), \( \overline{I} \)

\[
\begin{align*}
&= \int_0^\infty \frac{x^{p-1}}{x^2 + 1} \, dx + e^{i\pi} \int_0^\infty \frac{x^{p-1} e^{2\pi i (p-1)}}{x^2 e^{2\pi i} + 1} \, dx \\
&= \int_0^\infty \frac{x^{p-1}}{x^2 + 1} \left( 1 - e^{-2\pi i (p-1)} \right) \, dx \\
&= \int_0^\infty \frac{x^{p-1}}{x^2 + 1} e^{i\pi (p-1)} \left( e^{-i\pi (p-1)} - e^{-i\pi (p-1)} \right) \\
&= -2i e^{i\pi (p-1)} \int_0^\infty \frac{x^{p-1}}{x^2 + 1} \sin \left( \pi (p-1) \right) \\
\end{align*}
\]

which must be equal to

\[
\begin{align*}
&= -\frac{2\pi i}{\sin(\pi/2)} e^{i\pi (p-1)} \frac{\sin \frac{\pi}{2} (p-1)}{\sin(\pi/2)} \\
\end{align*}
\]

we cancel \(-2i e^{i\pi (p-1)}\) from both sides and use \(e^{i\pi} = -1\) to get

\[
\begin{align*}
\int_0^\infty \frac{x^{p-1}}{x^2 + 1} \, dx & = \Pi \frac{\sin \left( \pi (p-1)/2 \right)}{\sin \left( \pi (p-1) \right)} \\
& = \frac{\Pi}{2} \frac{1}{\cos \left( \pi (p-1)/2 \right)} \\
\end{align*}
\]

where we used \(\sin 2a = \sin a \cos a\).
Analytic continuation

Let $f_1(z)$ and $f_2(z)$ be analytic functions in a region $D$.

If they agree

(a) in a neighborhood of a point in $D$

(b) a line segment in $D$

(c) or at an accumulation point belonging to $D$

then in any one of these 3 cases they agree

consider $f_1(z) - f_2(z)$. Since this function is analytic in $z$ its zeroes must be isolated, unless the function is identically zero. For cases

(a) or (c) means the zeroes are not isolated - i.e. we can find zeroes arbitrarily close to another one.
Assume $f(z)$ is analytic in an open region $D$. Let $γ(t)$ be a path from $z_1$ to $z_2$ in $D$.

Since $f(z)$ is analytic at $z_1$, there is a circle about $z_1$ where the Taylor series converges.

Inside of this circle the function is determined by the coefficients of the Taylor series.

We cover the curve by a finite set of circles in $D$ with the property that the center of the $(n+1)$th circle is in the interior of the $n$th circle.

The Taylor coefficients of the $n$th circle define $f(z)$ at the center of the $(n+1)$st circle. This can be used to compute the Taylor coefficients for expansion around the center of the $(n+1)$st circle.

In this way, the Taylor coefficients around the starting point uniquely determine $f(z)$ in all of $D$.

This process is called **analytic continuation**.
More generally if $f_1(z)$ and $f_2(z)$ are analytic in $D_1$ and $D_2$ (open regions) and $f_1(z) = f_2(z)$ on $D_1 \cap D_2$, then $f_1(z)$ and $f_2(z)$ have analytic continuation to a single function

$$h(z) = \begin{cases} f_1(z) & z \in D_1 \\ f_2(z) & z \in D_2 \setminus D_1 \end{cases}$$

In this case $D_1 \cap D_2$ is an open set.

Next assume $f_1(z)$ is analytic in $D_1$ and $f_2$ is analytic in $D_2$ and $D_1$ and $D_2$ have a common boundary, where

1. $f_1(z) = f_2(z)$ on the common boundary

2. $f_1(z)$ and $f_2(z)$ are continuous on the boundary

(here we do not assume that $f(z)$ is analytic on the boundary)
Theorem: Assume

1. $f(z)$ is analytic in a region $D$
2. Assume the boundary of $D$ contains part of the real line.
3. Assume $f(z)$ is defined on the real boundary, real, and continuous on the boundary.

Then $f(z)$ has an analytic continuation to $D^*$ given by $g(z) = f^*(z^*)$.

First note $g(z)$ is analytic in $D^*$

\[
\frac{dg}{dz} = \lim_{\Delta z \to 0} \frac{g(z + \Delta z) - g(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{f^*(z^* + \Delta z^*) - f^*(z^*)}{\Delta z^*} = \lim_{\Delta z \to 0} \left( \frac{f(z + \Delta z) - f(z)}{\Delta z} \right)^*.
\]

This shows that $g(z)$ is analytic in $D^*$. 
\[ f(z) = \sum a_n z^n \]
\[ g(z) = \sum a_n^* z^{x_n} \]

since both functions are real on the common line segment, agree on that segment, and are continuous. Then by the previous theorem they are both part of the same analytic function

\[ h(z) = \begin{cases} f(z) & z \in D \\ g(z) & z \in D^* \end{cases} \]

Dispersion relations

* assume \( h(z) \) is analytic except for a branch cut from 0 to \( a \)
* \( |h(z)| \to 0 \) as \( |z| \to \infty \)
* \( h(z) \) is real for \( x < a \)

Then

\[ h(z) = \frac{1}{\pi} \int_0^\infty \frac{\text{Im} h(x+iy)}{x-z} \, dy \]

This means \( h(z) \) is determined by the discontinuity across the branch cut.
let $\gamma$ be the curve

For $z$ in the region enclosed by $\gamma$

$$h(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{h(z')}{z'-z} \, dz'$$

There are 4 parts to the integral

$$h(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{h(z')}{z'-z} \, dz' =$$

$$\frac{1}{2\pi i} \int_{0}^{\alpha} \frac{h(x+ie)}{x-z} \, dx + \frac{1}{2\pi i} \int_{0}^{2\pi} \frac{h(Re^{i\phi})}{Re^{i\phi}-z} \, iRe^{i\phi} \, d\phi +$$

$$\frac{1}{2\pi i} \int_{\alpha}^{\infty} \frac{h(x-ie)}{x-z} \, dx + \frac{1}{2\pi i} \int_{\frac{2\pi}{3}}^{\pi} \frac{h(Re^{i\phi})}{Re^{i\phi}-z} \, iRe^{i\phi} \, d\phi$$

The second integral vanishes in the limit $R \to \infty$ by Darboux theorem.

Because $|h(z)| \leq C |h(z)| \frac{1}{R} |z| > R$

The fourth integral also vanishes as $R \to \infty$ by Darboux theorem.
What remains is

\[ h(z) = \frac{1}{2\pi i} \int_0^\alpha \left( \frac{h(x+i\epsilon)}{x-z} - \frac{h(x-i\epsilon)}{x-z} \right) \, dx \]

The difference is the discontinuity across the branch cut.

Because \( h(-x+i\epsilon) \) is real and \( \text{Schwarz reflection principle} \) give

\[ h(x+i\epsilon) = h^*(x+i\epsilon) \]

\[ h(z) = \frac{1}{2\pi i} \int_0^\alpha \frac{h(x+i\epsilon)-h^*(x+i\epsilon)}{x-z} \, dx - \frac{1}{2\pi} \int_0^\alpha \text{Im} \, h(x+i\epsilon) \, dx \]

Assume \( f(z) \) is a bounded meromorphic function with simple poles at \( \{Z_1, \ldots, Z_n\} \).

Let \( \gamma \) be a closed counterclockwise curve around \( Z_1, \ldots, Z_n \).

\[ \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-z_0} = f(z) + 2 \pi i \frac{d}{dz} \left( \frac{1}{z-z_0} \right) 
\]

\[ f(z) + 2 \pi i \frac{1}{z-z_0} \]
where \( r_1 \) is the residue of \( f(z) \) at \( z_i 

\[
\begin{align*}
\oint \frac{f(z')}{z(z'-z)} \, dz' = \frac{1}{2\pi i} \oint \frac{f(z')}{z'-z} \left( \frac{1}{z'-z} - \frac{1}{z} \right) \, dz' \\
- 2 \, r_i \left( \frac{1}{z_i - z} - \frac{1}{z_i} \right)
\end{align*}
\]

Note

\[
\frac{1}{z_i - z} - \frac{1}{z_i} = \frac{z_i + z - z_i}{z_i(z_i - z)} = \frac{z}{z_i(z_i - z)}
\]

This means

\[
\left| \frac{1}{2\pi i} \oint \frac{f(z')}{z(z'-z)} \, dz' \right| \sim \frac{R}{R^2} \quad \text{for bounded } f(z)
\]

as the radius of the curve \( R \to \infty \)

\[
\begin{align*}
\oint \frac{f(z')}{z(z'-z)} \, dz' = f(z) - 2 \, r_i \left( \frac{z - z_i^2}{z_i(z_i - z)} \right) \\
\oint \frac{f(z')}{z(z'-z)} \, dz' = f(z) + \sum_{i=1}^{n} \frac{z}{(z-z_i)(z_i - z)}
\end{align*}
\]

which shows that this function has a simple form.
Next consider

\[
\frac{1}{2\pi i} \int_{C} \frac{1}{g(z)} \frac{df}{dz} \, dz
\]

where

1. C is a closed contour where \( g(z) \) is analytic and \( n = 0 \).

2. Assume \( f(z) \) has \( N \) zeroes \( z_1, \ldots, z_N \) and \( M \) poles \( \bar{z}_1, \ldots, \bar{z}_M \) in the interior of \( C \).

Here we assume that the zeroes and poles are of some finite order, but not necessarily 1.

About each pole or zero \( f(z) \) has a Laurent expansion of the form

\[
f(z) = \sum_{n=n_0}^{\infty} a_n (z-z_i)^n
\]

where \( n_0 \) could be negative

\[
f'(z) = \sum_{n=n_0}^{\infty} n a_n (z-z_i)^{n-1}
\]

If we consider \( \frac{f'(z)}{f(z)} \) near \( z_i \):

\[
\frac{\sum_{n=n_0}^{\infty} a_n (z-z_i)^{n-1}}{\sum_{m=n_0}^{\infty} a_m (z-z_i)^m} = \frac{n_0}{z-z_i} \left( 1 + \sum_{m=n}^{\infty} \frac{a_m (z-z_i)^{m-n}}{a_n (z-z_i)^n} \right)
\]
the residue of each pole \( a_n \) is \( n_0 \).

This means

\[
\frac{1}{2\pi i} \oint \frac{1}{f(z)} \frac{df}{dz} \, dz = \sum n_+ - \sum n_-
\]

\( n_+ \) is the order of the \( i^{th} \) zero
\( n_- \) is the order of the \( i^{th} \) pole.

If \( f(z) \) is a polynomial of degree \( N \),

\[
\frac{1}{2\pi i} \oint \frac{1}{P_n(z)} \frac{dP_n}{dz} \, dz = \frac{1}{2\pi i} \oint \frac{a_N N z^{N-1}}{(1 + \frac{a_N}{a_{n-1}} z^{N-1})^{1/2} + \ldots} \, dz
\]

in \( \text{as } z \to \infty \) what survives

\[
\frac{1}{2\pi i} \oint \frac{dz}{z} = N \quad \text{for a circle at } \infty
\]

This means

\[
N = \sum n_+ = \# \text{ roots of } P_n
\]

which means that a polynomial of degree \( N \) has \( N \) complex roots.

This is the fundamental theorem of algebra.