1) Factor \( x^2 + 1 = 0 \)
   \( x^2 + 1 = (x - i)(x + i) \)

2) **Complex numbers:**
   \( z = a + bi \) \( a, b \) real

3) \( |z| = \sqrt{a^2 + b^2} \) modulus
   \( \phi \) argument
   \( \cos \phi = \frac{x}{|z|} \quad \sin \phi = \frac{y}{|z|} \quad z = |z|(\cos \phi + i \sin \phi) \)

4) **algebra**
   \( z_1, z_2, z_1 \cdot z_2, \frac{1}{z}, \frac{1}{z_1} - z_2 \)
   \( z = x + iy \quad z^* = x - iy \quad z = \sqrt{zz^*} \)
   \( |z_1 + z_2| < |z_1| + |z_2| \)

5) **complex functions of complex variables**
   \( P(z) = \sum_{n=0}^{\infty} c_n z^n \) polynomials

Sums defined by series:
   \( e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^{x+iy} \)

**convergence, Cauchy sequences:**
   \( |f(z) - \sum_{i=1}^{N} f_n(z)| < \epsilon \)

**related functions**
   \( \sin z, \cos z, \sinh z, \cosh z \)
   \( \sin(i2z) = isinh z \)
   \( \cos(i2z) = cosh z \)
   etc
all trig function identities that hold for real values hold in complex \(z\).

\[
Z = 121e^{i\phi}
\]

\[
sin Z = \frac{1}{2i}(e^{iZ} - e^{-iZ}) \quad \sinh Z = \frac{e^Z - e^{-Z}}{2}
\]

\[
cos Z = \frac{1}{2}(e^{iZ} + e^{-iZ}) \quad \cosh Z = \frac{e^Z + e^{-Z}}{2}
\]

\[\text{6) } \ln Z \text{ def}\]

\[
Z = e^{\ln Z} \quad \ln Z = \ln|Z| + i(\phi + 2\pi n) \quad Z = 121e^{i\phi}
\]

multivalued function

\[
\ln Z_1^{z_1} = z_2 \ln Z_1
\]

\[
\ln (Z_1Z_2) = \ln Z_1 + \ln Z_2, \text{ etc.}
\]

\[\text{6) complex derivative}\]

\[
f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}
\]

* limit must exist
* limit must be unique

\[
\Delta z = 1 \Delta z e^{i\phi} \quad \text{limit must be independent of } \phi
\]

\[\text{7) consequences}\]

\[
f(z) = U(x, y) + iV(x, y)
\]

\[
\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}
\]

\[
f(x, y) = f(z, \bar{z}) \rightarrow \frac{\partial f}{\partial z} = 0
\]
\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u = 0 \quad \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) v = 0
\]

\[\nabla u \cdot \nabla v = 0\]

\[\frac{dz}{dz} = \frac{\partial x}{\partial x} + i \frac{\partial y}{\partial x} = 0\]

\[\frac{dz}{dz} = \frac{\partial x}{\partial y} + i \frac{\partial y}{\partial y} = 0\]

(6) \(f(z)\) analytic at \(z_0\).

1. \(f(z)\) exists

2. \(f(z)\) single valued in a neighborhood of \(z_0\).

(6) Complex integration

\[\oint_{C} f(z) \, dz = \int_{\lambda_1} f(z) \left( \frac{\partial x}{\partial \lambda} + i \frac{\partial y}{\partial \lambda} \right) \, d\lambda\]

Circle of radius \(r\)

\[X(\theta) = r \cos \theta + i r \sin \theta \quad 0 \leq \theta < 2\pi\]

defined as limit of Riemann sums

\[= \int_{\lambda_1} (f_x + f_y) \, d\lambda + i \int_{\lambda_1} (f_x + f_y) \, d\lambda\]

(6) Singular points - points in complex plane where \(f(z)\) is not analytic.
(i) Conformal mappings

\[ x'(xy) + iy'(xy) = \varphi(z) \quad : \quad (x'y') \rightarrow (x'y') \]

if \( \frac{dz}{d^2}(z) \neq 0 \)

This can be inverted in a neighborhood of \( z \).

Conformal mappings preserve angles.

(ii) Homeomorphic transformations

\[ z' = \frac{az+b}{cz+d} \quad ad - bc \neq 0 \]

Transformations form a group

Isomorphic to \( SL(2, \mathbb{R}) \)

Preserve angles

Map circles to circles

generated by

1. Inversion \( z \rightarrow \frac{1}{z} \)
2. Scale transformation \( z \rightarrow cz \)
3. Translation \( z \rightarrow z + c \)

(iii) Cauchy's Integral Theorem

If \( \varphi(z) \) analytic in \( \Omega \)

\[ \text{closed curve in } \Omega \]

\[ \oint \varphi(z) \, dz = 0 \]

(iv) Poincaré \( \oint \varphi(z) \, dz \leq L \cdot \max \| \varphi(z) \| \)
5 Cauchy integral representation

\[ \frac{1}{2\pi i} \oint \frac{f(z')}{z - z'} dz' = (N_{\text{cw}} - N_{\text{ccw}}) f(z) \]

\( N_{\text{cw}} \) = # counterclockwise loops around \( z \),
\( N_{\text{ccw}} \) = # clockwise loops around \( z \)

6 Consequence

all derivatives of \( f(z) \) exist

\[ \frac{d^n f}{dz^n} = \frac{n!}{2\pi i} \oint \frac{f(z')}{(z - z')^{n+1}} dz' \]

Taylor series about \( z \) converges in any disk in \( \text{region of analyticity} \)

\( f(z) \) can't have a local maximum in \( \mathbb{R} \)

zeros of \( f(z) \) must be isolated

\( |f(z)| < \epsilon f(z) \) entire \( \rightarrow \) \( f(z) = \text{constant} \)

\( |f(z)| < \epsilon |z|^n f(z) \) entire \( \rightarrow \) polynomial degree \( \leq N \)

7 Example of integral representation

\[ f(z) = \oint K(z, z') g(z') dz' \text{ and in } z \text{ for } z' \in C \]

8 \[ f(z) = \bar{z} \overline{f_n(z)} \text{ unit conv.} \]

analytic

\( \bar{z}' = 2 \bar{z} \)
Laurent series about a singular point in a simply connected region

\[ f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \]

Laurent series converges uniformly in any sub-disk

\[ a_n = \frac{1}{2\pi i} \oint_{|z|=\epsilon} \frac{f(z)}{(z^n - a_n)} \, dz \]

\[ b_n = \frac{1}{2\pi i} \oint_{|z|=\epsilon} \frac{f(z)}{(z^n - a_n)} \, dz \]

Singularities

1. Poles (order n)
2. Essential singularities
3. Weierstrass test: \( f(z) \) gets arbitrarily close to any complex number in any neighborhood of \( z_0 \)

Poles are isolated

Morera's Theorem

\[ \oint_{C} f(z) \, dz = 0 \text{ for any closed curve } C \]

Converge to 0 or \( \infty \)

\[ f(z) = \sum a_n z^n \quad \text{as} \quad \text{conv} = \text{analytic} \]

Poisson

\[ U(r, \theta) = \frac{1}{2\pi} \int_{0}^{2\pi} U(r', \theta') \frac{R^2 - r^2}{R^2 - 2r R \cos(\theta - \theta')} \, d\theta' \]
Residue Theorem

\[ \text{Res} \left( f(z) \right) = \frac{1}{2\pi i} \oint_C f(z') \, dz' \]

C curve around singular point counterclockwise

\[ \text{Res} \left( f(z) \right) = b_1 \text{ in Laurent series counterclockwise} \]

Simple pole: \[ f(z) = \frac{g(z)}{z-z_0}, \quad \text{Res} f(z) = g(z_0) \]

if \[ g(z_0) \neq 0 \]

Order \( n \) pole: \[ f(z) = \frac{g(z)}{(z-z_0)^n}, \quad g(z_0) \neq 0 \]

\[ \text{Res} \left( f(z) \right) = \frac{1}{(n-1)!} \frac{d^{n-1} g}{dz^{n-1}} (z_0) \]

\[ \oint_C f(z) \, dz = 2\pi i \sum \text{Res} (f(z)) \]

Joukowsky Lemma

\[ I = \oint_C e^{iaz} f(z) \, dz \]

|f(z)| < E(R, \theta) \text{ independent of } z \text{ in } \Omega \]

\( \Omega \cup \text{unbounded in } a > 0 \)

\( \Omega \cup \text{unbounded in } \cup \Omega \text{ unbounded in } a < 0 \)
Examples of integrals

\[
\sum_{n=-\infty}^{\infty} f(n) = -\frac{1}{2\pi}\text{Res}(f(z)\cot(\pi z)) \quad (2f(z) \to 0 \text{ as } z \to \infty)
\]

Principal value

Branch points

Branch cuts

\[ z - \frac{1}{2} = \infty \]

Multivalued function

\[ z^m, \quad m \in \mathbb{Z} \]

Integrating multivalued functions

Analytic continuation

\[ f_1, f_2 \text{ are on } \text{line, open set} \]

Analy - contin on common boundary

Schwarz - real on real line - contin

Dispersion relation

\[ \text{Im } f(z) \text{ on cut } \Rightarrow f(z) \]

Boundary men.

\[ f(z^2) = f_1(\omega) + \sum \frac{z \gamma_i}{z_i(z - 2)} \]

\[ \int_{2\pi i} f = 2(n_+ - n_-) \]

\[ \text{fund thm algebra} \]
\[ \int_{c} e^{wf(z)} g(z) \, dz \]

An independent choice of \( c \) with \( c \) on \( z \) path.

Ramme function