Last time

Linear Functionals

These are continuous linear maps from a vector space $V$ to the complex (real) numbers:

$L(\mid v \rangle) = \text{complex}$

$L(\mid v_1 \rangle + \alpha \mid v_2 \rangle) = L(\mid v_1 \rangle) + \alpha L(\mid v_2 \rangle)$

Continuous means that if $\mid v_n \rangle \to \mid v \rangle$
then
$L(\mid v_n \rangle) \to L(\mid v \rangle)$

Is complex numbers

The space of continuous linear functionals on a vector space is another vector space called the dual space to $V$.

Examples of continuous linear functionals

1. $V =$ space of degree 2 polynomials
$L(p) = p(\frac{1}{2})$

2. $V =$ space of $2 \times 2$ complex matrices
$L(A) = \text{Tr}(BA) = \sum_{i=1}^{2} B_{i3} A_{3i}$ for $B \in V$.

3. $V =$ Hilbert space
$L(\mid v \rangle) = \langle \omega \mid v \rangle$ for $\omega \in \text{V}$.
For Hilbert spaces there is a 1-1 correspondence between vectors and linear functions
\[ |w\rangle \rightarrow \langle w | \]
\[ L(|v\rangle) = \langle w | v \rangle \]

\( m \) defines a continuous linear functional
\[ L(|v_1\rangle + \alpha |v_2\rangle) = \langle w | v_1 \rangle + \alpha \langle w | v_2 \rangle \]

\[ |v_n\rangle \rightarrow |v \rangle \]
\[ |L(|v_0\rangle) - L(|v\rangle)| = |\langle w | v_0 \rangle - \langle w | v \rangle| \]
\[ = |\langle w | v_0 - v \rangle| \leq ||w|| ||v_0 - v|| \rightarrow 0 \]
as long as \( ||v_0|| < \infty \).

In some case one is trying to find a solution to a system of equations - but the solution is not a vector in the space \( V \) however it might be a solution on a bigger vector space that has \( V \) as a subspace.

Example
Let \( |v_n\rangle \) be an infinite collection of orthonormal vectors on an inner product space \( V \)
\[ \langle v_m | v_n \rangle = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} = \delta_{mn} \]
Let \( V \) be the set of vectors of the form
\[
\|v\rangle = \sum_{n=0}^{\infty} c_n |v_n\rangle
\]
\[
\langle v | v \rangle = \sum_{m,n=0}^{\infty} c_m^* c_n \langle v_m | v_n \rangle = \sum_{m,n}^{\infty} c_m^* c_n S_{mn}
\]
\[
= \sum_{n=0}^{\infty} |c_n|^2 < \infty
\]

(1)

Next we define 3 other vector spaces
\[
S = \sum_{n=0}^{\infty} d_n |v_n\rangle
\]
where
\[
\sum_{n=0}^{\infty} |d_n|^2 (n^2+1) < \infty
\]

(2)

and
\[
S^* = \sum_{n=0}^{\infty} e_n |v_n\rangle
\]
\[
\sum_{n=0}^{\infty} |e_n|^2 \frac{1}{(n^2+1)} < \infty
\]

(3)

Clearly (2) is more restrictive than (1), while (3) is less restrictive than (1).

We have
\[
S \subset V \subset S^*
\]
Note that if \( |V \rangle \in S \) and \( |W \rangle \in S \),

\[
\langle W | V \rangle = \left| \sum_{n=0}^{\infty} \frac{e_n}{\sqrt{n+1}} \langle n | V \rangle \right| =
\]

\[
\sum_{n=0}^{\infty} \frac{e_n}{\sqrt{n+1}} \sqrt{(n+1)^2} d^n =
\]

\[
|\tilde{\omega}\rangle = \sum_{n=0}^{\infty} e^n \frac{1}{\sqrt{n+1}} = \langle \tilde{\omega} | \tilde{\omega} \rangle < \infty
\]

\[
|\tilde{\nu}\rangle = \sum_{n=0}^{\infty} d^n \frac{1}{\sqrt{n+1}} = \langle \tilde{\nu} | \tilde{\nu} \rangle < \infty
\]

\[
\langle \tilde{\omega} | \tilde{\nu} \rangle = \sqrt{2} \sum_{n=0}^{\infty} \frac{1}{n+1} \sqrt{2} d^n (n+1)^2 < \infty
\]

so while \( \langle \omega \rangle \) is not defined on every vector in \( V \), it is defined on every vector in \( S \).

With this type of structure we can look for solutions of a system of equations in \( S \).

Example: quantum mechanics

\[
E = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2m}
\]

\[
e^{ikx/m} \rightarrow \frac{\hbar^2}{2m} \frac{d^2}{dx^2} e^{ikx/m} = \frac{\hbar^2}{2m} e^{ikx/m}
\]

The usual Hilbert space is square integrable functions on \( [-\infty, \infty] \)

\[
\int_{-\infty}^{\infty} (e^{ikx/m}) \, (ikx/m) \, dx = \infty
\]
if we consider
\[ \int_{-\infty}^{\infty} \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \frac{\hbar^2}{2m} \right) e^{-ikx/\hbar} f(x) \, dx = 0 \]

In a large class of function \( f(x) \), then we call \( e^{-ikx/\hbar} \) a weak solution of the equation
\[ \left( -\frac{\hbar^2}{2m} \frac{d^2}{dy^2} - \frac{\hbar^2}{2m} \right) g(y) = 0 \]

This will be discussed more formally later, this method of enlarging the space of linear functions which lies in the space that they act on is important in physics - the relevant subject is distribution theory.

Linear operators

A linear operator \( A \) is a mapping from a subspace \( D_1 \) of a vector space \( V_1 \) to a subspace \( R_1 \) of a vector space \( V_2 \) satisfying
\[ A(\lambda v_1 + \alpha v_2) = \lambda A(v_1) + \alpha A(v_2) \]

The sum on the left is in \( D_1 \subset V_1 \) while the sum on the right is in \( R_2 \subset V_2 \).
The space of linear operators from $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is a vector space.

\[(A_1 + A_2)\langle v_1 \rangle = A_1\langle v_1 \rangle + A_2\langle v_1 \rangle\]

\[A_1(\alpha \langle v_1 \rangle) = \alpha A_1\langle v_1 \rangle\]

It is easy to check that addition of linear operators and multiplication of operators by scalars satisfy all of the axioms of a vector space.

For example

\[\lambda (A_1 + A_2)\langle v \rangle = \lambda (A_1\langle v \rangle + A_2\langle v \rangle) = \lambda A_1\langle v \rangle + \lambda A_2\langle v \rangle = (\lambda A_1)\langle v \rangle + (\lambda A_2)\langle v \rangle\]

etc.

If $A_1: \mathbb{R}^n \rightarrow \mathbb{R}^m$

and $A_2: \mathbb{R}^m \rightarrow \mathbb{R}^n$

then we can define $A_2 A_1: \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

\[(A_2 A_1)\langle v \rangle = A_2(A_1\langle v \rangle)\]
If $V_1$ and $V_2$ are normed linear spaces then we define the operator norm

$$\|A\| = \sup_{\|v\|_1 \neq 0} \frac{\|Av\|_2}{\|v\|_1} = \sup_{\|v\|_2 = 1} \|Av\|_2$$

Here $\sup$ means least upper bound.

Note that

$$\|Av\|_1 \leq \sup_{\|v\|_2} \frac{\|Av\|_1}{\|v\|_2} \cdot \|v\|_2 = \|A\| \cdot \|v\|_2,$$

this means that if $\|v_n\| - \|v\|_1 < \epsilon$

$$\|Av_n\| - \|Av\|_1 \leq \|A\| \cdot \|v_n\| - \|v\|_2 = \|A\| \cdot \epsilon$$

This means that $A: V_1 \to V_2$ is a continuous linear map if the operator norm of $A$ is finite.

Continuous linear operators are called bounded linear operators.
A linear operator $A: V_1 \to V_2$ is 
\underline{onto} if \underline{in} every vector $|v_2\rangle \in V_2$
\underline{there is at least one} $v_1 \rangle$ in $V_1$
\underline{with the property}

$|v_2\rangle = A |v_1\rangle$

A linear operator $A: V_1 \to V_2$ is 
\underline{one-to-one} if $|v_1\rangle \neq |v_1\rangle'$ implies

$A |v_1\rangle \neq A |v_1\rangle'$

A linear operator $A: V_1 \to V_2$ that is \underline{1-1 and onto has an inverse}

$B = A^{-1}: V_2 \to V_1$

Let $|v_2\rangle$ be any vector in $V_2$. \underline{Since $A$ is onto we can write}

$|v_2\rangle = A |v_1\rangle$

for some vector $|v_1\rangle$ in $V_1$. \underline{If there is another vector $|v_1\rangle' = |v_1\rangle$ such that}

$|v_2\rangle = A |v_1\rangle' = A |v_1\rangle$

then $A$ will not be \underline{1-1}. \underline{Therefore}

$|v_1\rangle$ \underline{is unique}. Define

$B |v_2\rangle = |v_1\rangle$. \underline{.}
Since $\{v_i\}$ is arbitrary this defines $B$ for all vectors in $V_2$

\[
ABv_2 = A\{v_i\} = Iv_2
\]

\[
B\{v_i\} = Bv_2 = Iv_1
\]

For linear operators from $\{V_1 \rightarrow V_1\}$, in addition to addition and scalar multiplication of operators, if $D_1 V_1$, we also have the product of operators

\[
\{v_i\}
\]

\[
A\{v_i\} = Iv_1
\]

\[
A^2\{v_i\} = A\{v_i\} = A^2 Iv_1
\]

This can be extended to polynomials in $A$

\[
P(A) v = \sum_{n=0}^{\infty} c_n A^n v
\]

If $V$ is a normed linear vector space and $A$ is a continuous map from $V \rightarrow V$ then
\[ \| A^2 \| \leq \sup \frac{\| A^2 \|}{\| A \|} = \sup \left( \frac{\| A^2 \|}{\| A \|}, \frac{\| A \|}{\| A \|} \right) \leq \sup \left( \frac{\| A^2 \|}{\| A \|} \right), \sup \left( \frac{\| A \|}{\| A \|} \right) \leq \sup \frac{\| A^2 \|}{\| A \|}, \sup \frac{\| A \|}{\| A \|} = \| \| A \| \| \cdot \| A \| \|. \]

This can be continued by induction:

\[ \| A^n \| \leq \| \| A \| \| \cdot \| A \| \|. \]

We can use this to construct more complicated functions of \( A \):

\[ e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}, \]

\[ \| e^A \| = \| \sum_{n=0}^{\infty} \frac{A^n}{n!} \| \leq \sum_{n=0}^{\infty} \frac{\| A^n \|}{n!} \leq \sum_{n=0}^{\infty} \frac{\| A \|^n}{n!} \| A \|^{\| A \| n} = e \]

This means that if \( \| A \| < 1 \) this series converges uniformly.
example:
matrices
\[ e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!} \]
we need to check
\[ \| M \|_\infty < \infty \]
we will learn how to compute \( \| M \|_\infty \) later, but it is clear that if \( M \) acts on a unit vector, the length of the resulting vector can be more than \( (\text{dim of matrix}) \times (\text{absolute value of largest matrix element}) \).

example
linear differential equation:
\[ \frac{dx_1}{dt} = a_{11} x_1 + a_{12} x_2 \]
\[ \frac{dx_2}{dt} = a_{21} x_1 + a_{22} x_2 \]
\[ \frac{d^2 x}{dt^2} = A^n \tilde{x} \]
\[ \tilde{x}(t) = \sum_{n=1}^{\infty} \frac{A^n \tilde{x}(0)}{n!} t^n = e^{At} \tilde{x}(0) \]
this series converges as long as
\[ \| A \|_\infty < \infty \]
In addition to $e^A$, we can construct:

\[ \sinh A = \sum_{n=0}^{\infty} \frac{A^{2n+1}}{(2n+1)!} \]
\[ \cosh A = \sum_{n=0}^{\infty} \frac{A^{2n}}{(2n)!} \]
\[ \sin A = \sum_{n=0}^{\infty} (-1)^n \frac{A^{2n+1}}{(2n+1)!} \]
\[ \cos A = \sum_{n=0}^{\infty} (-1)^n \frac{A^{2n}}{(2n)!} \]

This can be done for any convergent series.

We can also consider $e^{Az}$ where $z$ is a complex number. The series

\[ e^{Az} = \sum_{n=0}^{\infty} \frac{A^n z^n}{n!} \]

converges in norm since $||Az|| \leq ||z|| ||A|| < \infty$.

In this case,

\[ \frac{d}{dz} e^{Az} = \lim_{\Delta z \to 0} \frac{e^{A(z+\Delta z)} - e^{Az}}{\Delta z} \]

has a complex derivative. It is an open map valued analytic function.
Next consider the operator

\[(Z I - A)\]

where \(A\) is a linear operator.

If this operator has an inverse \(Z_i\) and \(Z_j\) we have

\[(Z_i I - A) - (Z_j I - A) = (Z_i - Z_j) I\]

multiply on one side by \((Z_i I - A)^\dagger\)
and on the other side by \((Z_j I - A)^\dagger\)
to get

\[(Z_j I - A)^\dagger - (Z_i I - A)^\dagger = (Z_j I - A)^\dagger (Z_i - Z_j) (Z_i I - A)^\dagger\]

this is usually written

\[(Z_j I - A)^\dagger = (Z_i I - A)^\dagger + (Z_i I - A)^\dagger (Z_i - Z_j) (Z_i I - A)^\dagger\]

If we choose \(Z_j\) so \(\| (Z_i I - A)^\dagger (Z_i - Z_j) \| < 1\)
we can solve this by iteration

\[(Z_j I - A)^\dagger = (Z_i I - A)^\dagger + \sum_{n=1}^{\infty} ((Z_i I - A)^\dagger (Z_i - Z_j)) (Z_i I - A)^\dagger\]

by choice of \(Z_j\) is series converges
in particular

\[(Z_i I - A)^\dagger = (Z_i I - A)^\dagger + \sum_{n=1}^{\infty} (Z_i I - A)^\dagger (Z_i - Z_j) (Z_i I - A)^\dagger\]

is an analytic function of \(Z_\infty\)

\[\| (Z_i I - A)^\dagger (Z_i - Z_j) \| < 1\]
$R(z) = (zI-A)^{-1}$ is called the resolvent of $A$. The equation

$$R(z_2) = R(z_1) + R(z_1)(z_1-z_2)R(z_2)$$

is called the limit resolvent equation.

Other types of operators:

1. An operator $A$ is antilinear if

$$A(\alpha|v_1\rangle + \alpha'|v_2\rangle) = A(\alpha|v_1\rangle) + \alpha^* A(|v_2\rangle)$$

Antilinear operators are not linear. The most important example of an antilinear operator is time reversal.

2. Let $A_1, A_2$ be linear operators on a vector space $V$

$$[A_1, A_2] = A_1 A_2 - A_2 A_1$$

is called the commutator of $A_1$ and $A_2$.

When the commutator is non-zero then $A_1 A_2 \neq A_2 A_1$. 
The anticommutator of 2 linear operators on \( V \) is defined by

\[ [A_1, A_2] = A_1 A_2 + A_2 A_1. \]

Inverses: If \( A_1 \) and \( A_2 \) act on \( V \) and both have inverses, then

\[ (A_1 A_2)^{-1} = A_2^{-1} A_1^{-1}. \]

Check

\[ (A_1 A_2)^{-1} A_1 A_2 = A_2^{-1} A_1^{-1} A_1 A_2 = A_2^{-1} A_2 = I. \]

Adjoint operator: If \( A \) is a linear operator on an inner product space, the adjoint \( A^* \) of \( A \) is defined by

\[ \langle V_1 | A^* V_2 \rangle = \langle A V_1 | V_2 \rangle = \langle V_2 | A V_1 \rangle^*. \]
**Hermitian Operator:**

An operator $A$ on an inner product space $V$ is Hermitian if

$$A^* = A$$

This means

$$\langle Av, w \rangle = \langle v, Aw \rangle$$

**Unitary Operator:**

An operator $A$ on an inner product space $V$ is unitary if

$$A^* = A^{-1}$$

**Projection Operator:**

An operator $P$ on an inner product space is a projection operator if

$$P = P^2$$

$$P^2 = P$$
An operator $A$ on an inner product space is normal if

\[ [A^*, A] = 0 \]

An operator on a vector space $V$ is nilpotent if, for some finite $n$,

\[ A^n = 0 \]

example $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

remark and properties

1. \((A_1, A_2)^* = A_2^* A_1^* \)

   \[ \langle V | A_1 A_2 | V_2 \rangle = \]
   \[ \langle A_2^* V | A_1 | V_2 \rangle = \]
   \[ \langle A_2^* A_1 V | V_2 \rangle = \]
   \[ \langle (A_1 A_2)^* V | V_2 \rangle \]

   since this holds for all $|V_1>, |V_2>$

\((A_1, A_2)^* = A_2^* A_1^* \)

2. If $A = A^*$, $B = B^*$ in general

   \[ AB \neq (AB)^* = B^* A^* = BA \]

   then these to be equal

\[ [A, B] = 0 \]
A linear operator on a vector space is antihermitean if
\[ A^* = -A \]

Note: if \( A \) is Hermitian, \( iA \)
is antihermitean
\[ (iA^*) = -iA \]

An operator \( A \) on an inner product space is positive if
\[ \langle v|A|v \rangle \geq 0 \]
for all \( |v \rangle \) (this must be real and non-negative).

* If \( A \) is positive
\[ \langle v|A|v \rangle = \langle v|A^*|v \rangle^* = \]
\[ \langle Av|v \rangle = \langle A^*v|v \rangle \]

Next, let \( |v \rangle = |v_1\rangle + \alpha |v_2\rangle \)
\[ \langle v_1 + \alpha v_2 | A (v_1 + \alpha v_2) \rangle = \]
\[ \langle A (v_1 + \alpha v_2) | (v_1 + \alpha v_1) \rangle = \]
\[ \langle v_1 | Av_1 \rangle + |\alpha|^2 \langle v_2 | Av_2 \rangle + \alpha^* \langle v_2 | Av_1 \rangle + \alpha \langle v_1 | Av_2 \rangle = \]
\[ \langle Av_1 | v_1 \rangle + |\alpha|^2 \langle Av_2 | v_2 \rangle + \alpha^* \langle Av_2 | v_1 \rangle + \alpha \langle Av_1 | v_2 \rangle \]
terms 1, 5, and 2 and 6 cancel

letting $d$ be 1 or $i$ give

$\langle v_1 A v_1 \rangle = \langle A v_1 v_1 \rangle$
$\langle v_1 A v_1 \rangle = \langle A v_1 v_1 \rangle$

This means all positive operators are Hermitian.

If $P$ is a projection operator

$\langle v_1 P v_1 \rangle = \langle v_1 P^2 v_1 \rangle = \langle P v_1 P v_1 \rangle$
$\langle P v_1 P v_1 \rangle \geq 0$

This shows that all projection operators are positive.

If $A$ is a linear operator and

$A v_1 = \lambda v_1$

1. the vector $v_1$ is called an eigenvector of $A$
2. the number $\lambda$ is an eigenvalue of $A$
1. If $A = A^\dagger$, the eigenvalues of $A$ are real.

   $A |v\rangle = \lambda |v\rangle$

   $\langle v | A |v\rangle = \lambda \langle v | v\rangle$

   $\langle v | A |v\rangle^\dagger = \lambda^* \langle v | v\rangle$  

   $\langle A | v \rangle = \lambda \langle v |v\rangle$

   $\langle A | v \rangle = \lambda^* \langle v |v\rangle$

   $\langle v | A - A^\dagger |v\rangle = (\lambda - \lambda^*) \langle v |v\rangle$

   This requires $\lambda = \lambda^*$ unless $|v\rangle = 0$

2. If $A = A^\dagger$ and $\lambda_1 \neq \lambda_2$ are eigenvalues of $A$, then $\langle v_1 | v_2 \rangle = 0$

   $\langle v_2 | A |v_1\rangle = \langle v_2 | \lambda_1 |v_1\rangle = \lambda_1 \langle v_2 |v_1\rangle$

   $\langle A | v_2 \rangle$

   $\langle v_1 | A |v_2\rangle^\dagger = (\lambda_2 \langle v_1 |v_2\rangle)^\dagger = \lambda_2^* \langle v_1 |v_2\rangle$

   $\lambda_2 \langle v_2 |v_1\rangle$

   Since $\lambda_2 = \lambda_2^*$, we have

   $(\lambda_1 - \lambda_2) \langle v_2 |v_1\rangle = 0$

   Since $\lambda_1 \neq \lambda_2$, $\langle v_1 |v_2\rangle = 0$
If $A$ is \underline{normal}, then

$$A = \frac{1}{2} (A + A^t) + \frac{i}{2} (i (A^t - A))$$

$$B = \frac{1}{2} (A + A^t) = B^t$$

$$C = \frac{i}{2} (A^t - A) = C^t$$

$$A = B + iC$$

where since $[A, A^t] = 0$, $[B, C] = 0$.

So $A$ is a linear combination of 2 commuting linear operators.

If $U$ is unitary and

$$\langle U | U \rangle = \lambda |U\rangle$$

then

1. $|\lambda| = 1$
2. $\lambda_1 \neq \lambda_2$, $\langle U_1 | U_2 \rangle = 0$

$$\langle V_1 | V_2 \rangle = 1 = \langle V_1 | U_1 U_2 \rangle = \langle U_1 U_1 U_2 \rangle = \langle U_1 U_1 U_2 \rangle = |\lambda_1| \lambda_2 \langle V_1 | U_2 \rangle$$

So

$$(1 - \lambda_1 \lambda_2) \langle V_1 | V_2 \rangle = 0 \quad \Rightarrow \quad \langle V_1 | V_2 \rangle = 0$$

$$(\lambda_1 - \lambda_2) \langle V_1 | V_3 \rangle = 0 \quad \lambda_1 \neq \lambda_2 \quad \langle V_1 | V_3 \rangle = 0$$