Last time

x A is positive if $\langle \nu_1 A \nu_1 \rangle \geq 0$

x Generalized Schwarz inequality $A \geq 0$

$|\langle \nu_1 A \nu_2 \rangle|^2 \leq \langle \nu_1 A \nu_1 \rangle \langle \nu_2 A \nu_2 \rangle$

x Square root theorem

Every positive operator has a positive square root

x If $\|A\| = x$ we and define $A' = \frac{1}{2x} A$ which implies that

$\|A'\| < 1$. $B'^2 = A'$ then $B = \sqrt{2x} B'$ satisfies $B^2 = A$ so it is sufficient to show that $A$ has a positive square root for $\|A\| < 1$

x In seeking a solution to

$B^2 = A$

let

$A = 1 - C$

$B = 1 - D$
This gives the following relation between $C$ and $D$.

$$B^2 = (1 - D)^2 = 1 - 2D + D^2 = A = 1 - C$$

canceling 1 gives

$$D^2 + C = 2D$$

or

$$D = \frac{1}{2} \left( C + D^2 \right)$$

By definition $C = 1 - A$ so

$$<v_1|c|v_1> = 1 - <v_1|a|v_1> \quad \text{for} \quad <v_1|v> = 1$$

In addition

$$<v_1|a|v_1> \leq \|v_1\|^{1/2} \|a\|^{1/2} \leq \|v_1\|^{1/4} \|a\|^{1/4} \|v_1\|^{1/4} \|a\|^{1/4}$$

$$= 1 \cdot 1 \cdot \|a\|^{1/2} < 1$$

since

$$0 \leq <v_1|a|v_1> < 1$$

$$0 \leq 1 - <v_1|c|v_1> < 1$$

which means $C$ is a positive operator with $0 \leq <v_1|c|v_1> \leq 1$.
we try to solve this by iteration

\[ D_0 = \frac{1}{2} C \]
\[ D_1 = \frac{1}{2} (C + D_0) = \frac{1}{2} (C + \frac{1}{2} C^2) = \frac{1}{2} C + \frac{1}{8} C^2 \]
\[ D_2 = \frac{1}{2} (C + D_1^2) = \frac{1}{2} (C + \frac{1}{2} C^2 + \frac{1}{8} C^3 + \frac{1}{64} C^4) \]

in general

\[ D_n = \frac{1}{2} (C + D_{n-1}^2) \]

(1)

in addition

\[ D_n - D_{n-1} = \frac{1}{2} (C + D_{n-1}^2) - \frac{1}{2} (C + D_{n-1}) \]
\[ = \frac{1}{2} (D_{n-1} - D_{n-2}) (D_{n-1} + D_{n-2}) \]

(2)

Note that if \( C \) is positive then \( C^n \) is positive.

Even \( n = 2m \) \( \langle v \mid C^{n} v \rangle = \langle C^{m} v \mid C^{m} v \rangle = \| C^{m} v \|^{2} \geq 0 \)
Odd \( n = 2m+1 \) \( \langle v \mid C^{n} v \rangle = \langle C^{m} v \mid C \cdot C^{m} v \rangle \geq 0 \)

It follows that

(1) \( D_n \) is a sum of positive operators with positive coefficients. We showed this for \( D_0, D_1, D_2 \); if it holds for \( D_{n-1} \), it holds for \( D_n \) by (1).

(2) \( D_1 - D_0 = \frac{1}{8} C^2 > 0 \)

If \( D_{n-1} - D_{n-2} \) is a sum of positive operators with positive coefficients, (2) implies this is true for \( D_{n-1} - D_{n-2} \).
\[ \| D \|_1 \leq \frac{1}{2} \| C \|_1 \leq \frac{1}{2} \| (1 - A) \|_1 \leq \frac{1}{2} (\| C \|_1 + \| A \|_1) < 1 \]

if \( \| D_{n-1} \|_1 < 1 \) then

\[ \| D_n \|_1 = \| \frac{1}{2} (C + D_{n-1}^2) \|_1 \]

\[ \leq \frac{1}{2} \| C \|_1 + \frac{1}{2} \| D_{n-1} \|_1 \cdot \| D_{n-1} \|_1 \]

\[ \leq \frac{1}{2} \cdot \frac{1}{2} < 1 \]

We also have

\[ \langle v | D_{n-1} v \rangle \leq \| v \|_2 \| D_{n-1} \|_1 \leq \| v \|_2 \| D_{n-1} \|_1 \| D_{n-1} \|_1 \leq 1 \cdot 1 \]

since \( \langle v | D_{n-1} v \rangle \geq 0 \) we have

\[ 0 \leq \langle v | D_{n-1} v \rangle \leq 1 \]

It means that the numbers \( \langle v | D_{n} v \rangle \) are an increasing sequence of positive numbers bounded by 1.

Define \( \langle v | D_{n} v \rangle \) as the least upper bound of this sequence of numbers.

This means \( \langle v | D_{n} v \rangle \) converges to \( \langle v | D_{n} v \rangle \).
Next consider
\[
\langle v + aw | D_n | v + aw \rangle =
\langle v | D_n | v \rangle + i |\alpha|^2 \langle w | D_n | w \rangle
\]
\[
\quad + \alpha \langle v | D_n | w \rangle + \alpha^* \langle w | D_n | v \rangle
\]
\[
\quad a = 1
\]
\[
\langle v + w | D_n (v + w) \rangle - \langle v | D_n | v \rangle - \langle w | D_n | w \rangle
\]
\[
\quad = \langle v | D_n | w \rangle + \langle w | D_n | v \rangle
\]  \qquad (1)
\[
\quad a = i
\]
\[
\langle v+iw | D_n (v+iw) \rangle - \langle v | D_n | v \rangle - \langle w | D_n | w \rangle
\]
\[
\quad = i \langle w | D_n | v \rangle - i \langle v | D_n | w \rangle
\]
\[
\quad - i \langle v+iw | D_n (v+iw) \rangle - i \langle v | D_n | v \rangle - i \langle w | D_n | w \rangle
\]
\[
\quad = - \langle w | D_n | v \rangle + \langle v | D_n | w \rangle
\]  \qquad (2)

Adding (1) and (2) give:
\[
\langle v | D_n | w \rangle = \frac{1}{2} \left( \langle v + w | D_n (v + w) \rangle + i \langle v+iw | D_n (v+iw) \rangle - \langle v | D_n | v \rangle - i \langle v+iw | D_n (v+iw) \rangle - \langle w | D_n | w \rangle - i \langle w | D_n | w \rangle \right)
\]

All of the terms on the right converge to $D_n \rightarrow D$ which give:
\[
\langle v | D | w \rangle = \frac{1}{2} \left( \langle v + w | D (v + w) \rangle + i \langle v+iw | D (v+iw) \rangle - (1+i) \langle v | D | v \rangle + \langle w | D | w \rangle \right)
\]
Consider 
\[ \| (D_n - D) u \|^2 = \langle (D_n - D) u, (D_n - D) u \rangle. \]

Using the generalized Schwarz inequality:
\[ \leq \langle v, (D_n - D) u \rangle \langle (D_n - D) u, (D_n - D) u \rangle \]
\[ = \langle v, (D_n - D) u \rangle \langle v, (D_n)^3 u \rangle \]
\[ \leq \langle v, (D_n - D) u \rangle \| (D_n - D) u \| \| (D_n)^3 u \| \]
\[ \leq \langle v, (D_n - D) u \rangle \| (D_n - D) u \| \| (D_n)^3 \| \]
\[ \leq \langle v, (D_n - D) u \rangle \delta \]

Since \( \langle v, (D_n - D) u \rangle \to 0 \), it follows that \( \| (D_n - D) u \| \to 0 \).

This means:

1. \( D_n \) converges strongly to \( D \)

2. \( A^{1/2} = B = I - D \)

3. Since \( \langle v, (D_n) u \rangle \leq 1 \), \( \langle v, (D) u \rangle \leq 1 \)
   which means
   \[ \langle v, (I - B) u \rangle = 1 - \langle v, (D) u \rangle \geq 0 \]
   so \( B = \sqrt{A} \) is positive.

4. Since \( D_n \) is a polynomial in \( A \)
   \[ A^{1/2} A^{1/2} = 0 \]
Polar decomposition

Assume $A$ is a bounded linear operator with a bounded inverse.

Then

1. $AA^T$ and $A^TA$ are positive

$$<u^TAA^Tv> = <A^TvA^Tu> \geq 0$$
$$<v^TA^TAu> = <A^TuA^Tv> \geq 0$$

2. $AA^T$ and $A^TA$ have positive square roots:

$$\sqrt{A^TA} \quad \sqrt{AA^T}$$

3. $\sqrt{A^TA}$ and $\sqrt{AA^T}$ have inverses:

$$[(A^TA)^{-1}]^T \sqrt{A^TA} = (A^TA)^{-1} (A^TA) = I$$
$$[(A^TA)^{-1}]^T \sqrt{AA^T} = (AA^T)^{-1} (AA^T) = I$$

Note that since $\sqrt{A^TA}$ is a senior in $AA^T$ it commutes with $A^TA$:

$$\left( \sqrt{A^TA} \right)^{-1} = (A^TA)^{-1} \sqrt{A^TA} = \sqrt{A^TA} \left( A^TA \right)^{-1}$$
$$\left( \sqrt{AA^T} \right)^{-1} = (AA^T)^{-1} \sqrt{AA^T} = \sqrt{AA^T} \left( AA^T \right)^{-1}$$
Result
\[ A = A (A^+A)^{-\frac{1}{2}} (A^+A)^{\frac{1}{2}} = (AA^+)(AA^+)^{-\frac{1}{2}} A \]

Note
\[ (A^+A)^{\frac{1}{2}} \geq 0 \quad (AA^+)^{\frac{1}{2}} \geq 0 \]

and
\[ (AA^+)^{-\frac{1}{2}} A (AA^+)^{\frac{1}{2}} \]
\[ = (AA^+)^{-\frac{1}{2}} (AA^+)(AA^+)^{-\frac{1}{2}} = (AA^+)(AA^+)^{-\frac{1}{2}} = I \]
\[ (AA^+)^{-\frac{1}{2}} (AA^+)^{\frac{1}{2}} = I \]
\[ (AA^+)(AA^+)^{-\frac{1}{2}} = I \]

This shows that any bounded invertible operator can be expressed as

\[ A = PU = U^P \]

where \( U \) and \( U' \) are unitary and 
\( P \) and \( P' \) are positive.
If $A$ is a linear operator then

Moore–Penrose generalized inverse of $M$ is an operator $A^+$ satisfying

$$A^+ A A^+ = A^+$$
$$A A^+ A = A$$

$$(A^+) (A A^+) = (A A^+)$$
$$(A^+ A ) (A^+ A ) = (A^+ A )$$
$$(A^+) = (A^+ A)$$

Clearly if $A$ has an inverse then $A$ satisfies these equations.

It turns out that even when $A$ does not have an inverse these equations have a unique solution.

Some programs that compute inverses of matrices give an answer even when the matrix has no inverse. The program usually gives the Moore generalized inverse.
constructive methods use an iteration of the form

\[ A_n^+ = \alpha A^+ \]

\[ A_n^+ = 2A_{n-1}^+ - A_{n-1}^+ AA_{n-1}^+ \]

where \( \alpha \) is a sufficiently small real number \( 0 < \alpha < \frac{2}{\|A\|^2} \).

It can be shown that the resulting series \( A_n^+ \) is a Cauchy sequence that converges to an operator satisfying the Penrose equations.
Finite dimensional vector spaces

Def: Linearly independent vectors
A collection \( \{ l_{an} \}_{n=1}^{\infty} \) are linearly independent if the only solution to the equation

\[
\sum_{n=1}^{\infty} c_n l_{an} = 0
\]

is \( c_n = 0 \) for \( n = 1, \ldots, M \)

Def: Spanning sets
A collection of vectors \( \{ l_{an} \}_{n=1}^{\infty} \) span a vector space if any vector \( l_{v} \in V \) can be expressed as a linear combination of the \( l_{an} \)

\[
V = \sum_{n=1}^{\infty} c_n l_{an}
\]

Theorem: Assume \( V \) is spanned by a set of \( N \) vectors. If \( \{ l_{an} \}_{n=1}^{K} \) are linearly independent then \( K \leq N \)

Let \( \{ l_{bn} \}_{n=1}^{N} \) be the spanning set.

\[
l_{am} = \sum_{n=1}^{N} c_{mn} l_{bn} \quad \text{(def of spanning)}
\]
Consider

\[ |a_1\rangle = \sum c_{1m} l_{bm} \]

at least one of the \( c_{1m} \) must be 0 - we can relabel the \( l_{bm} \) so \( c_{11} = 0 \) then we can write

\[ |b_1\rangle = \frac{1}{c_{11}} |a_1\rangle - \sum_{m=2} c_{1m} l_{bm} \]

Next consider

\[ |a_2\rangle = \sum c_{2m} l_{bm} \]

\[ = c_{21} |a_1\rangle + \sum_{m=2} c_{2m} l_{bm} \]

(this is because \( |a_1\rangle \) is a linear combination of \( |a_1\rangle \) and \( |l_{bm}\rangle \) in \( m \geq 2 \)

since \( |a_2\rangle \) is independent of \( |a_1\rangle \) we must have one of the \( c_{2m} \neq 0 \)

(if not \( |a_2\rangle = c_{21} |a_1\rangle = 0 \) which contradicts independence)

\( c_{22} \neq 0 \) so relabel \( |b_2\rangle \sim |l_{b2}\rangle \) so \( b_2 \rightarrow b_1 \)

\[ |a_2\rangle = c_{21} |a_1\rangle + c_{22} |b_2\rangle + \sum_{m=3} c_{2m} l_{bm} \]

\[ |b_2\rangle = -\frac{1}{c_{22}} c_{21} |a_1\rangle + \frac{1}{c_{11}} |a_1\rangle - \sum_{m=3} \frac{c_{2m}}{c_{22}} l_{bm} \]

We can keep repeating this

\[ |b_k\rangle = \sum_{n=1} d_{kn} |a_n\rangle + \sum_{m=k+1} c_{kn} l_{bm} \]
If we continue to $k = n$

$$|a_k> = \sum \delta_{Ik} |a_k>$$

this means $|a_1>, \ldots, |a_n>$ is a spanning set

$$|a_{n+1}> = \sum_{m=1}^{n} c_{n+m} |a_m>$$

which means

$$|a_{n+n}| = \sum_{m=1}^{n} c_{n+m} |a_m> = 0$$

has a non-zero solution. This proves

$k \leq n$.

**Definition:** The dimension of a vector space is the maximum number of linearly independent vectors.

**Definition:** A basis for a vector space $V$ is a linearly independent set of vectors that span $V$.

- $\text{span} \quad N_g \geq \dim V$
- Independent $\quad N_h \leq \dim V$
- # basis vectors $= \dim of V$
Is \|a\| is a basis and

\[ |c\rangle = \sum \lambda_n a_n \langle a_n | a\rangle = \sum \lambda_n a_n \langle a_n | a\rangle \]

\[ \Delta = |c\rangle - |c\rangle = \sum (\lambda_n - \lambda_n) a_n \langle a_n | a\rangle \]

since the \|a_n\| are linearly independent

\[ \lambda_n - \lambda_n = 0 \quad \text{so} \quad \lambda_n = \lambda_n \]

This means that the coefficients relating a vector to a basis are unique.

The numbers \( \lambda_n \) are called the coordinates of the vector \(|c\rangle\) in the basis \{\(a_n\}\}.

Note: If we have 2 distinct bases \( \{\|v_n\| \}_{n=1}^\infty \) and \( \{\|w_n\| \}_{n=1}^\infty \), the coordinates of \(|c\rangle\) depend on the basis:

\[ |c\rangle = \sum \lambda_n |v_n\rangle = \sum \lambda_n |w_n\rangle \]

\[ \lambda_n \neq \lambda_n \quad |v_n\rangle \neq |w_n\rangle \]
given a basis \{v_n\}, vector space operations are replaced as operations on coordinates.

1. Vector addition

\[ l_a = \sum a_n v_n \]
\[ l_b = \sum b_n v_n \]
\[ l_a + l_b = \sum (a_n + b_n) v_n = \sum c_n v_n \]
\[ c_n = a_n + b_n \]

2. Multiplication by scalar:

\[ \alpha l_a = \alpha \sum a_n v_n = \sum \alpha a_n v_n = \sum c_n v_n \]
\[ c_n = \alpha a_n \]

3. Linear operation \( A \)

\[ A l_v = \sum_{m=1}^n l_m v_m a_{mn} \]
\[ l_b = A (\sum l_n v_n) \]
\[ = \sum A l_n v_n \]
\[ = \sum l_m v_m a_{mn} b_n \]
\[ = \sum l_m v_m d_n \]
\[ d_m = \sum_{n=1}^n a_{mn} b_n \]
the numbers $a_{mn}$ are matrix elements of the operator $A$ in the basis $|v_n>$.

Let $a_{mn}$ and $b_{mn}$ be matrix elements of operators $A$ and $B$ in the basis $|v_m>$. Then

$$A |v_m> = \sum_{n} a_{nm} |v_n>$$

$$B |v_m> = \sum_{n} b_{nm} |v_n>$$

$$\Rightarrow AB |v_m> = \sum_{n} a_{nm} \sum_{n} b_{nm} |v_k>$$

$$= \sum_{n} \sum_{k} a_{nm} b_{km} |v_k>$$

Thus if $A \cdot B = C$ then the matrix elements of these operators in the basis $|v_n>$ are related by

$$C_{mn} = \sum_{k=1}^{n} a_{nk} b_{km}$$

$C_{mn}$ is normally expressed as a square array of numbers where $m =$ row number, $n =$ column number.

$C_{11} = 1$
$C_{12} = 2$
$C_{21} = 3$
$C_{31} = 4$
These representations by matrices depend on the choice of basis.

The basis vectors only have to be a linearly independent spanning set.

\[ |a\rangle = \sum |v_n\rangle a_n \]
\[ B|v_n\rangle = \sum |v_\lambda\rangle b_{\lambda n} \]
\[ B|a\rangle = \sum B|v_n\rangle a_n = \sum |v_\lambda\rangle b_{\lambda n} a_n \]

\[ l = B|a\rangle = \sum |v_\lambda\rangle c_{\lambda} \]

We have

\[ c_{\lambda} = \sum b_{\lambda n} a_n \]

\[
\begin{pmatrix}
  c_1 \\
  \vdots \\
  c_n
\end{pmatrix} =
\begin{pmatrix}
  b_{11} & b_{12} & \cdots & b_{1n} \\
  b_{21} & b_{22} & \cdots & b_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{n1} & b_{n2} & \cdots & b_{nn}
\end{pmatrix}
\begin{pmatrix}
  a_1 \\
  \vdots \\
  a_n
\end{pmatrix}
\]

The relation between \( c_{\lambda} \), \( b_{\lambda n} \) and \( a_n \) can be expressed in matrix form.
Note

\[ |a\rangle = \sum n |v_n\rangle a_n \]

\[ |b\rangle = \sum m |v_m\rangle b_m \]

\[ \langle a | b \rangle = \langle \sum n v_n | v_m \rangle b_m \]

\[ = \sum a_n^* \langle v_n | v_m \rangle b_m \]

In the general case the inner product is expressed in terms of

\[ V_{nm} = \langle v_n | v_m \rangle \]

\[ \langle a | b \rangle = \sum a_n^* V_{nm} b_m \]

while \( a_n \), \( b_m \), \( V_{nm} \) are all basis dependent, \( \langle a | b \rangle \) is not.

If

\[ A |v_n\rangle = \sum m |v_m\rangle A_m \]

then

\[ \langle v_k | A | v_n\rangle = \sum_m \langle v_k | v_m \rangle A_m = \sum_m V_{km} A_m \]

To find the coefficients we must solve this linear system for the \( A_m \). Formally

\[ A_m = \left( V^{-1} \right)_{mn} \langle v_k | A | v_m \rangle \]

when

\[ \sum_k V_{mk} \langle v_k | v_n \rangle = \delta_{mn} \]
Note that if $\{\nu_n\}$ is a basis then

\[ \nu_n = \sum_{m} a_{nm} \nu_m \]

\[ \langle \nu_m \nu_n \rangle = \delta_{nm} \]

This can be simplified if the basis vectors are orthonormal:

\[ \langle \nu_n \nu_m \rangle = \delta_{nm} \]

If the basis vectors are not orthonormal, they can be replaced by a new basis that is orthonormal.

1. Assume $\{\nu_n\}$ is an arbitrary basis.
2. Replace $\nu_n$ by $\nu_n' = \nu_n / \sqrt{\langle \nu_n \nu_n \rangle}$
   
   In this case $\langle \nu_n' \nu_n' \rangle = 1$.
3. Let $\nu_1' = \nu_1' = \nu_1$.
4. Let $\nu_2' = (\nu_2' - \langle \nu_2' \nu_1' \rangle \nu_1')$.

\[ \langle \nu_1' \nu_2' \rangle = \langle \nu_1' \nu_2' \rangle - \langle \nu_1' \nu_1' \rangle \langle \nu_2' \nu_1' \rangle = 0 \]

Let $\nu_2' = \nu_2' / \sqrt{\langle \nu_2' \nu_2' \rangle}$.
by induction assume \( \langle \omega_m | \omega_n \rangle = s_{mn} \), \( m, n < k \)

define
\[
|w_k\rangle = \sum_{n=1}^{k-1} \langle \omega_n | v_k \rangle |\omega_n\rangle
\]

This is not 0 because \( |v_k\rangle \) is independent of \( |v_1\rangle, |v_{k-1}\rangle \) and the \( |\omega_n\rangle \) are linear combinations of these vectors.

\[
\langle \omega_0 | w_k \rangle = \langle \omega_0 | v_k \rangle - \langle \omega_k | v_k \rangle = 0, \quad k < K
\]

define \( |w_k\rangle = |w_k\rangle / \sqrt{\langle w_k | w_k \rangle} \)

This can be repeated until \( k = N \), the \( \{ |w_k\rangle \} \) are orthonormal, so they are independent:

\[
\sum c_k |w_k\rangle = 0
\]

\[
\langle w_m | \sum c_k |w_k\rangle \rangle = c_m = 0
\]

and since any \( |v_k\rangle \) can be expressed in terms of the \( |\omega_n\rangle \), they are a basis.

For an orthonormal basis

\[
A |\omega_n\rangle = \sum |\omega_m\rangle a_{mn}
\]

\[
\langle \omega_k | A |\omega_n\rangle = \sum_{m=1}^{N} \langle \omega_k | \omega_m \rangle a_{mn} =
\]

\[
= \sum_{m=1}^{N} s_{km} a_{mn} = \delta_{mn}
\]
This means $\alpha_{mn} = \langle \psi_m | A | \psi_n \rangle$
if $\{|\psi_n\rangle\}$ is an orthonormal basis.

The method in constructing an orthonormal basis from a general basis is called the Gram-Schmidt method.

For vectors:

$$|a\rangle = \sum \alpha_n |\omega_n\rangle$$

$$\langle \omega_m | a \rangle = \sum \langle \omega_m | \omega_n \rangle \alpha_n = \alpha_m$$

$$|a\rangle = \sum \lambda_n |\omega_n\rangle \langle \omega_n | a \rangle$$

for orthonormal vectors.

Adjoint of a matrix

Consider

$$|b\rangle = \sum a_{nm} |\psi_n\rangle$$

$$\langle c | (A | b \rangle = \sum c^*_k \langle \psi_k | A | \psi_n \rangle a_{nm} b_m =$$

$$\langle A | c \rangle = \langle b | A^* c \rangle =$$

$$\sum (b^*_k \langle \psi_k | \psi_n \rangle a_{nm} c_m)^*_m =$$

$$\sum b^*_k \langle \psi_k | \psi_n \rangle a_{nm} c_m^*$$
\[ \sum C^x_R \langle \nu_n | \nu_n \rangle a_{nm} b_m = \]
\[ \sum C^x_R a_{RR} \langle \nu_e | \nu_m \rangle b_m \]

Comparing these expressions:

\[ \sum \langle \nu_n | \nu_n \rangle a_{nm} = \sum a_{er} \langle \nu_e | \nu_m \rangle \]

\[ a_{nm} = V_{nr} a_{er} V_{em} \]

where \[ V_{em} = \langle \nu_e | \nu_m \rangle \]

This is in a general basis.

For an orthonormal basis:

\[ a_{nm} = \langle \nu_n | a^+ A | \nu_m \rangle = a^T_{mn} \]

\[ = \langle \nu_m | A^T | \nu_n \rangle \]

The matrix elements of \( A^T \) in an orthonormal basis are the complex conjugate transpose of the matrix elements of \( A \).
define

\[ \mathcal{E}_{n_1 n_2} = \begin{cases} 1 & \text{if } n_1 = 1, n_2 = 2, n_3 = N \\ 0 & \text{if } n_i = n_j \text{ for some } i \neq j \\ \text{is completely antisymmetric} & \end{cases} \]

the determinant of \( A \) is an orthonormal basis \( \{ \mathbf{u}_n \} \) is defined by

\[
\det A = \sum_{n_1} a_{n_1, n_3} a_{n_2, n_3} \mathcal{E}_{n_1 n_2} \mathcal{E}_{n_3} 
\]

Later we will show that \( \det A \) is independent of basis, but so far it looks basis dependent.

Each row represents components of vectors

\[
\begin{align*}
\mathbf{a}_1, & \quad \mathbf{a}_2, & \quad \mathbf{a}_3, & \quad \cdot \cdot \cdot \\
\mathbf{a}_{n_1}, & \quad \mathbf{a}_{n_2}, & \quad \mathbf{a}_{n_3}, & \quad \cdot \cdot \cdot
\end{align*}
\]

If one of these rows are a linear combination of the other rows, e.g.

\[
\det A = 0 \quad a_{n_1}, \quad a_{n_2}, \quad a_{n_3}, \quad \mathcal{E}_{n_1 n_2} \quad \mathcal{E}_{n_3} 
\]

\[
\sum_{k} c_k \mathcal{E}_{n_1 n_k}
\]
assume that $c_i \neq 0$

$$\det A = (c_1 a_{i_1} a_{i_2} \cdots a_{i_N} + \cdots)$$

\[\underbrace{\epsilon_{n_1 n_k n_\nu}}_{-\epsilon_{n_k n_1 n_\nu}}\]

We relabel dummy indices

$$a_{i_1} \rightarrow a_{n_k}$$

$$a_{n_k} \rightarrow a_{i_1}$$

$$\det A = - (c_1 a_{i_1} a_{i_2} a_{n_k} a_{n_\nu} + \cdots)$$

$$\underbrace{\epsilon_{n_1 n_k n_\nu}}_{\epsilon_{n_1 n_k n_\nu}}$$

In each non-zero $c_R$ the contribution to the determinant is 0.

$$\det A = 0$$ is the rows of $A$ are linearly dependent

Consider

$$\frac{\partial}{\partial a_{i_1}} \det A$$

the removes the $i^{th}$ row and $j^{th}$ column of $a_{i_1}$ from the expression for the determinant. If we multiply by $a_{i_1}$

$$\det A = 2 a_{i_1} \frac{\partial}{\partial a_{i_1}} \det A$$
If instead we replace the $i^{th}$ row by the $k^{th}$ row, we now have dependent rows.

$$\sum_j A_{mj} \frac{\partial}{\partial a_{ij}} \det A = 0 \quad m \neq i$$

In general,

$$A_{mj} \frac{\partial}{\partial a_{ij}} \det A = \delta_{mi} \det A$$

It follows that if $\det A \neq 0$, then:

$$\delta_{mj} = \frac{1}{\det A} \sum_j A_{mj} \frac{\partial}{\partial a_{ij}} \det A$$

This means

$$(\tilde{A}^{-1})_{ij} = \frac{1}{\det A} \frac{\partial}{\partial a_{1i}} \det A$$

The matrix $\frac{\partial \det A}{\partial a_{ij}} = C_{ij}$ is called the cofactor matrix of $A$.

$C_{ij}$ is always defined — this means $A$ has an inverse if $\det A \neq 0$. 

...
\[ A \rightarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \]

\[ \det A = a_{11}a_{22} - a_{12}a_{21} \]

\[ C_{11} = \frac{\partial}{\partial a_{11}} \det A = a_{22} \quad C_{12} = \frac{\partial}{\partial a_{21}} \det A = -a_{12} \]

\[ C_{21} = \frac{\partial}{\partial a_{12}} \det A = a_{11} \quad C_{22} = \frac{\partial}{\partial a_{22}} \det A = -a_{21} \]

\[
\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{22} - a_{12} \\ -a_{21} - a_{11} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} =
\]

\[
\begin{pmatrix} a_{11}a_{22} - a_{12}a_{21} & a_{12}a_{22} - a_{12}a_{21} \\ -a_{21}a_{11} + a_{21}a_{11} & a_{21}a_{22} + a_{21}a_{11} \end{pmatrix} = \]

\[ \det A \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

which gives

\[ A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} - a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \]