Lecture 22

Last time
(1) $A > 0 \implies \sqrt{A} > 0$
(2) $A = PU = UP'$
(3) $AA^+ = A$, $A^+A = A^+$

$(AA^+) = \text{orthogonal projection}$
$(A^+A) = \text{orthogonal projection}$

Finite dimensional vector space:

(1) independent vectors

$$\sum c_n |v_n\rangle = 0 \implies c_n = 0 \text{ all } n$$

(2) spanning set

$$|v\rangle = \sum c_n |v_n\rangle \quad \text{for any } |v\rangle \in V$$

(3) $N_{\text{span}} \geq N_{\text{indep}}$

(4) Dimension $V$

$= \text{maximal } \# \text{ indep vectors}$

$N_{\text{indep}} \leq \dim(V)$

(5) Basis: indep vectors that span $V$

$\# = N_{\text{basis}}$
If $N_0$ is the dimension of $V$ then

$N_0 \geq N_{\text{basis}}$ since basis vectors are independent.

We also have

$N_{\text{basis}} \geq N_0$

since the basis is a spanning set.

$N_0 \leq N_{\text{basis}} \leq N_0$

So the number of basis elements is equal to the dimension of the space.

representation:

If $\{|v_n\rangle\}_{n=1}^N$ is a basis in an $N$ dimensional vector space

$|v\rangle = \sum_{n=1}^{N} c_n |v_n\rangle$

where the $c_n$ are complex coefficients.
The coefficients are called the components of the vector \( |v> \) in the basis \( \{ |v_n> \} \).

1. The coefficients of \(|v>\) in the basis \(\{ |v_n>\}\) are unique

\[ |v> = \sum_{n=1}^{N} |v_n> c_n = \sum_{n=1}^{N} |v_n> d_n \]

\[ 0 = \sum_{n=1}^{N} |v_n> (c_n - d_n) \]

Since the \(|v_n>\) are linearly independent

\[ 0 = c_n - d_n \quad \text{so} \quad c_n = d_n \]

2. If \(|v>\) is expressed in two different bases \(|v_n>, |w_n>\), the components of \(|v>\) in these different bases are not the same

\[ |v> = \sum_{n=1}^{N} |v_n> c_n = \sum_{n=1}^{N} |w_n> d_n \]
(3) Calculating the components of \( |V\rangle \) in the vector space \( V \)

\[
|V\rangle = \sum_i C_i |V_i\rangle
\]

To solve for the \( C_i \) take any basis \( \{ |V_i\rangle \} \) not necessarily \( |V\rangle \) and solve the system of linear equations for \( C_i \)

\[
\sum_{n=1}^{N} \langle W_m | V_n \rangle C_n = \delta_{mn}
\]

This is a set of \( N \) equations in \( N \) unknowns \( C_n \)

(4) It is useful to fix a basis and use coordinates in the basis to represent the vector

\[
|V\rangle = \sum_i C_i |V_i\rangle \rightarrow \begin{pmatrix} C_1 \\ \vdots \end{pmatrix}
\]

* \( C \) is not the vector - it allows one to construct the vector given the basis
Vector operations can be expressed in terms of components:

\[ |c> = |a> + |b> \]

\[ \sum |v_n> c_n = \sum |v_n> a_n + \sum |v_n> b_n \]

\[ \sum |v_n> (c_n - a_n - b_n) = 0 \]

Linear independence gives:

\[ c_n = a_n + b_n \]

\[ \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_m + b_m \end{pmatrix} \]

\[ \alpha |c> = \sum \alpha |v_n> a_n \]

\[ |c> = \sum |v_n> c_n \]

\[ \alpha |c> = \sum \alpha |v_n> c_n = \sum |v_n> (\alpha c_n) \]

which gives \( d_n = \alpha c_n \)

Let \( A \) be a linear operator and \( |v> \) be a vector with components \( c_n \)

\[ |v> = \sum |v_n> c_n \]
we also express \( \text{Al}_V \rangle \) in the basis of \( \text{Al}_V \rangle = \sum_{m} \text{Al}_V \rangle \text{A}_{mn} \text{C}_n \)

\[ \text{Al}_V \rangle = \sum_{n} \text{Al}_V \rangle \text{C}_n = \sum_{m} \text{Al}_V \rangle \text{A}_{mn} \text{C}_n = \sum_{m} \text{Al}_V \rangle \text{d}_m \]

so the components of \( \text{Al}_V \rangle \) are related to the components of \( \text{Al}_V \rangle \) by

\[ \text{d}_n = \sum_{m} \text{A}_{mn} \text{C}_m \]

To find the coefficients \( \text{A}_{mn} \) choose a basis \( \text{1}_W \rangle \) and solve

\[ \langle \text{w}_m | \text{A}_V \rangle \text{1}_V \rangle = \sum \langle \text{w}_m | \text{1}_V \rangle \text{A}_n \text{1}_n \]

in \( \text{A}_n \).
Given

\[ A |v_n> = \sum_{m} a_{mn} |v_m> \]
\[ B |v_n> = \sum_{m} b_{mn} |v_m> \]
\[ |v> = \sum_{n} |v_n> c_n \]

we can write

\[ AB |v> = \sum_{n} |v_n> d_n \]
\[ AB \sum_{n} |v_n> c_n \]
\[ A \sum_{mn} |v_m> b_{mn} c_n \]
\[ \sum_{k} |v_k> a_{km} b_{mn} c_n \]

equating coefficients

\[ d_n = \sum_{k,m} a_{km} b_{mn} c_m \]

we also have

\[ AB |v_n> = \sum_{k,m} a_{km} b_{mn} |v_m> \]

The coefficients \( a_{mn} \), \( b_{mn} \) are called matrix elements of \( A \) and \( B \) in the basis \( |v_n> \)
There are computational advantages to constructing orthonormal or bi-orthogonal bases.

1. \( \{ \mathbf{v}_n \} \) is an orthonormal basis if \( \langle \mathbf{v}_n \mid \mathbf{v}_m \rangle = \delta_{nm} \).

2. \( \{ \mathbf{v}_n \} \) and \( \{ \mathbf{w}_m \} \) are bi-orthogonal bases if \( \langle \mathbf{w}_n \mid \mathbf{v}_m \rangle = \delta_{nm} \).

**Theorem:** If \( \{ \mathbf{a}_n \} \) is an arbitrary basis we can use it to construct an orthonormal or bi-orthogonal basis.

1. Define \( \hat{\mathbf{a}}_n = \mathbf{a}_n / \sqrt{\langle \mathbf{a}_n \mid \mathbf{a}_n \rangle} \) for \( n = 1, N \).

2. Define \( \hat{\mathbf{v}}_1 = \hat{\mathbf{a}}_1 \).

3. Define \( \hat{\mathbf{v}}_2 = \mathbf{a}_2 - \hat{\mathbf{v}}_1 \langle \hat{\mathbf{v}}_1 \mid \mathbf{a}_2 \rangle \).

Then \( \langle \hat{\mathbf{v}}_1 \mid \hat{\mathbf{v}}_2 \rangle = \langle \hat{\mathbf{v}}_1 \mid \mathbf{a}_2 \rangle - \langle \hat{\mathbf{v}}_1 \mid \mathbf{a}_2 \rangle = 0 \).

4. Define \( \hat{\mathbf{v}}_2 = \mathbf{v}_2 / \sqrt{\langle \mathbf{v}_2 \mid \mathbf{v}_2 \rangle} \).
by induction assume we have defined \( \hat{\mathbf{v}}_i \ldots \hat{\mathbf{v}}_k \)

\[
\hat{\mathbf{v}}_{k+1} = \mathbf{v}_{k+1} - \sum_{i=1}^{k} \frac{\mathbf{v}^\dagger_i \mathbf{v}_{k+1} \mathbf{a}_{k+1}}{\mathbf{a}^\dagger_i \mathbf{a}_{k+1}}
\]

\[
\mathbf{a}_{k+1} = \mathbf{v}_{k+1} - \sum_{i=1}^{k} \frac{\mathbf{v}^\dagger_i \mathbf{v}_{k+1} \mathbf{a}_{k+1}}{\mathbf{a}^\dagger_i \mathbf{a}_{k+1}}
\]

This process can be continued until \( k = N \)

This is a basis since each \( \mathbf{v}_k \)

Involves \( \mathbf{a}_k \) with a non-zero coefficient plus \( \mathbf{a}_e \) with \( e < k \)

The construction is called the Gram-Schmidt construction

\[
\mathbf{v}_m \mathbf{v}_n = \delta_{mn}
\]

A similar construction can be used to replace \( \mathbf{v}_n \) by \( \hat{\mathbf{v}}_n \) that is biorthogonal with respect to a basis \( \mathbf{w}_n \)

(In this case the basis \( \mathbf{w}_n \) does not change.)
If \( \{ \psi_n \} \) is an orthonormal basis, then
\[
\psi > = \sum \psi_n \psi_n^\dagger \psi_n
\]
\[
\langle \psi_m | \psi > = \sum \langle \psi_m | \psi_n \rangle \psi_n = \sum \delta_{mn} \psi_n = \psi_n
\]
\(\psi_n^\dagger \psi_n = \langle \psi_n | \psi_n \rangle \)
\[\text{(orthonormal basis)}\]
\[
A \psi_n > = \sum \psi_m > a_{mn}
\]
\[
\langle \psi_m | A \psi_n > = \sum \psi_m \langle \psi_m | \psi_n \rangle a_{mn} = \sum \delta_{km} a_{mn}
\]
\[\text{(orthonormal basis)}\]

**Adjoint**

Recall
\[
\langle A^\dagger \psi | \omega > = \langle A^\dagger \psi | \omega > = \langle \omega | A^\dagger \psi \rangle^*
\]
\[
A^\dagger \psi_n > = \sum \psi_m > a^*_{mn}
\]
\[
\psi_n > = \sum \psi_n \psi_n^\dagger
\]
\[
\lambda > = \sum \lambda_n \psi_n
\]
\[
\langle \lambda | \omega > = \sum \langle \lambda_n | \omega \rangle \psi_n
\]
\[
\sum \psi_n \langle \psi_m | \psi_n \rangle = \delta_{mn} \psi_n = \psi_n
\]
\[
\sum \psi_n a_{mn} \langle \psi_n | \psi \rangle \psi_n = \sum \psi_n a_{mn} \langle \psi_n | \psi \rangle \psi_n = \sum \psi_n a_{mn} \psi_n^\dagger \psi_n = \sum \psi_n a_{mn} \psi_n
\]
\[
\sum \psi_n a_{mn} \langle \psi_n | \psi \rangle \psi_n = \sum \psi_n a_{mn} \psi_n
\]
Comparing these expressions, since \( v_n \) and \( \omega \) are arbitrary

\[
\sum_{vn} \langle v_n l | v_m \rangle A_{\omega n} = \sum A_{mn} \langle v_m | l v_n \rangle
\]

If the basis is orthogonal

\[
A_{\omega n}^+ = A_{\omega n}^* \quad \text{orthonormal basis}
\]

determinants

Let \( \varepsilon_{1 \cdots n} \) be defined by

\[
\varepsilon_{1 \cdots n} = \left\{ \begin{array}{l}
1 \quad n_i = i \quad i = 1 \cdots n \\
\text{completely antisymmetric on interchange of } n_i, \, n_j
\end{array} \right.
\]

\( \varepsilon_{123} = 1 \) (example)

\( \varepsilon_{123} = -1 \)

\( \varepsilon_{132} = 1 \)

\( \varepsilon_{132} = -1 \)

\( \varepsilon_{213} = 1 \)

\( \varepsilon_{231} = 1 \)
Let $a_{mn}$ be the matrix elements of $A$ in the basis $|n\rangle$.

Define

$$\det(A) = \sum E_i^{\mu_1} \cdots a_{i\mu_n}$$

Note that this definition looks like it is basis dependent. Later we will show that it does not depend on basis.

Consider vectors

$$|0_k\rangle = \sum |i\rangle|\mu\rangle$$

constructed out of the rows of $a_{mn}$.

If the $|i\rangle$ are not independent, this means

$$|0_k\rangle = \sum_{m=1}^{n} c_m |\mu_m\rangle$$

Using this in the expression for $\det A$ gives
Let $a_{mn}$ be the matrix elements of $A$ in the basis $|v_n\rangle$.

Define

$$\det(A) = \sum E_1^n \cdots E_n^n a_{1m} \cdots a_{mn}$$

Note that this definition looks like it is basis dependent. Later we will show that it does not depend on basis.

Consider vectors

$$|0\rangle = \sum |v_n\rangle \varphi_n$$

constructed out of the rows of $a_{mn}$.

If the $|v_k\rangle$ are not independent, this means

$$\varphi_k = \sum_{m+n} \alpha_m a_{me}$$

Using this in the expression for $\det A$ gives
\[
\det A = \sum_{n_1, n_2} \varepsilon_{n_1 n_2 n_3} a_{n_1} \cdot a_{n_2} \cdot \left( \sum_{k} a_{k R} a_{R k} \right) a_{k n} a_{n k}
\]

The values of \( k \) when \( C \to 0 \)

\[
\varepsilon_{n_1 n_2 n_3} a_{n_1} a_{n_2} a_{n_3} = a_{n_1 n_2} a_{n_1 n_3}
\]

\[
- \varepsilon_{n_1 n_2 n_3} a_{n_1} a_{n_2} a_{n_3} = a_{n_1 n_2} a_{n_1 n_3}
\]

We see that on interchanging any relabeling of \( n \), we get the identical quantities with \( \varepsilon \) sign.

\[
\varepsilon^{abc} (V_a V_b V_c) =
\varepsilon^{abc} (V_b V_c V_a) =
\varepsilon^{abc} (V_c V_a V_b) =
- \varepsilon^{bac} (V_b V_a V_c) =
- \varepsilon^{abc} (V_a V_b V_c)
\]

which means that
\[ \det A = 0 \text{ if any of the rows or columns of } a_{mn} \text{ are linearly dependent on the others.} \]

Consider:

\[ \frac{\partial}{\partial a_{ij}} \det A = \sum_{\mu} a_{\mu i} \frac{\partial}{\partial a_{i\mu}} \det A = \sum_{\mu} a_{\mu i} \cdot a_{in} \cdot a_{\mu n} \]

The derivative removes the circle quantity when \( n_i = \mu \).

We can put it back by multiplying by \( a_{ii} \):

\[ a_{ij} \frac{\partial}{\partial a_{ii}} \det A = \sum_{\mu} a_{\mu i} \cdot a_{i\mu} \cdot a_{ii} \cdot a_{\mu n} \]

Summing over \( \mu \) gives:

\[ \sum_{\mu} a_{ij} \frac{\partial}{\partial a_{ii}} \det A = \det A \]

On the other hand, using:

\[ \frac{\partial}{\partial a_{ij}} \det A \text{ replaces the } j \text{ th row by the } k \text{ th row.} \]
since that row already appears in the matrix we get

$$\sum_j a_{ij} \frac{\partial}{\partial q_{ij}} (\det A) = 0 \quad i \neq k$$

putting both equations together gives

$$\sum_j a_{ij} \frac{1}{\det A} \frac{\partial}{\partial q_{ij}} (\det A) = \delta_{ik}$$

since $\delta_{ik}$ are matrix elements of the identity in an orthonormal basis, this means if $\det A \neq 0$ then

$$A^{-1} = \frac{1}{\det A} \frac{\partial}{\partial q_{ij}} (\det A)$$

since $\det A$ is a product of component sof $A$ so $i \frac{\partial}{\partial q_{ij}} (\det A)$ always exist.

It is necessary and sufficient condition for $A$ to have an inverse is $\det A \neq 0$. 
The matrix

\[ C_{ij} = \frac{\partial}{\partial y_i^j} \det A \]

is called the cofactor matrix of \( A \)

\[ A_{ij} = \frac{C_{ij}}{\det A} \]

Example

\[ A = \begin{pmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{pmatrix} \]

\[ \det A = \sum_{i=1}^{n} E_{i2}^m a_{1m} a_{2n} \]

\[ = a_{11} a_{22} - a_{12} a_{21} \]

\[ \frac{\partial \det A}{\partial a_{11}} = a_{22} \quad \frac{\partial \det A}{\partial a_{12}} = a_{11} \]

\[ \frac{\partial \det A}{\partial a_{21}} = -a_{21} \quad \frac{\partial \det A}{\partial a_{22}} = -a_{22} \]

\[ C_{11} = a_{21} \quad C_{12} = -a_{21} \]

\[ C_{21} = a_{11} \quad C_{22} = -a_{12} \]

\[ \left( \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right) \left( \begin{array}{cc} a_{21} & -a_{12} \\ -a_{21} & a_{11} \end{array} \right) = \left( \begin{array}{cc} a_{11} a_{22} - a_{12} a_{21} & -a_{12} a_{11} + a_{11} a_{22} \\ a_{21} a_{22} - a_{22} a_{21} & a_{11} a_{22} - a_{12} a_{21} \end{array} \right) \]

\[ = \left( \begin{array}{cc} \det A & 0 \\ \partial \det A & 0 \end{array} \right) \]
Another useful way to express the determinant is in terms of permutations:

\[ \sigma \]

is a permutation of \( 1 \ldots N \) if

\[ \sigma(i) \neq \sigma(j) \]

for \( i \neq j \).

In 3 objects, there are 6 permutations:

\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 3 & 1
\end{pmatrix} \rightarrow \sigma(1) = 1 \quad \sigma(2) = 2 \quad \sigma(3) = 3
\]

\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 1 & 3
\end{pmatrix} \rightarrow \sigma(1) = 2 \quad \sigma(2) = 1 \quad \sigma(3) = 3
\]

\[
\begin{pmatrix}
1 & 2 & 3 \\
1 & 3 & 2
\end{pmatrix} \rightarrow \sigma(1) = 3 \quad \sigma(2) = 3 \quad \sigma(3) = 2
\]

\[
\begin{pmatrix}
1 & 2 & 3 \\
3 & 2 & 1
\end{pmatrix} \rightarrow \sigma(1) = 3 \quad \sigma(2) = 2 \quad \sigma(3) = 1
\]

\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 3 & 1
\end{pmatrix} \rightarrow \sigma(1) = 2 \quad \sigma(2) = 3 \quad \sigma(3) = 1
\]

\[
\begin{pmatrix}
1 & 2 & 3 \\
3 & 1 & 2
\end{pmatrix} \rightarrow \sigma(1) = 2 \quad \sigma(2) = 1 \quad \sigma(3) = 3
\]

These can be classified by the number of pairwise exchanges to get to the final order in:

1 2 3 \quad 1 3 2 \quad (1 \text{ transposition})

1 2 3 \quad 1 2 3 \rightarrow 1 3 2 \quad (2 \text{ transpositions})

(1 3) \quad (2 3)
Define \( |\sigma| = (-1)^{\text{# transpositions making } \sigma} \).

Homework: show that \( |\sigma| \) cannot be both even and odd.

If \( \mathcal{P} \) is the set of permutations on \( N \) objects, there are \( N! \) distinct permutations.

\[
\begin{align*}
N \text{ choices where to put } 1, \\
N-1 \text{ choices where to put 2,} \\
N-2 \text{ choices where to put 3,} \\
\vdots \\
1 \text{ choice where to put } N.
\end{align*}
\]

We can write

\[
\det A = \sum_{\sigma \in \mathcal{P}(N)} (-1)^{|\sigma|} A_{1 \sigma(1)} A_{2 \sigma(2)} \cdots A_{N \sigma(N)}
\]

This agrees with the previous definition since the coefficients are complete, symmetric, and \( |I| = 1 \).
Consider the determinant of $AB$ 
\[
\det (AB).
\]

The matrix elements of $AB$ are 
\[
(AB)_{ij} = \sum_{k} a_{ik} b_{kj}.
\]

Using these in the definition of $\det AB$ 
\[
\det AB = \sum_{\sigma} \sum_{\pi} a_{\sigma(1)} b_{\pi(1)} \cdots a_{\sigma(n)} b_{\pi(n)} (-1)^{\sigma \pi}.
\]

to get a non-zero result from 
\[
\sum_{\sigma} b_{\sigma(n)} (-1)^{\sigma n}
\]

the vectors $b_{\sigma(i)}$ have to be different so 
attn the sum over $\sigma$ the $\sigma$'s are all distinct. A given set of 
distinct $\sigma$'s can be expressed as 
\[
(\sigma_1 \cdots \sigma_n) = (\sigma'(1) \cdots \sigma'(n)).
\]

It follows that 
\[
\det AB = \sum_{\sigma \in S_n} a_{\sigma(1)} b_{\sigma(1)} \cdots a_{\sigma(n)} b_{\sigma(n)} (-1)^{\sigma(1)}
\]

Next introduce 
\[
1 = (-)^{\sigma_1'(1) + \sigma_2'(1)}
\]
\[
\det AB = \sum_{\sigma'} a_{i' j'} b_{\sigma' i' \sigma' j'} = \sum_{\sigma'} a_{i' j'} \prod_{k=1}^{n-1} b_{\sigma'(k)} \sigma'(k) = \prod_{k=1}^{n-1} b_{1 \sigma'(k)} \sigma'(k) \]

Next we recall the \( b_{\sigma' i' \sigma' j'} = b_{\sigma' i' \sigma' j'}(-1)^{\sigma'(k)} \) comes first and then \( \sigma' \).

\[
b_{1 \sigma'(1)} \quad b_{n \sigma'(n)}(-1)
\]

but

1. \( \sigma' \) another permutation
2. \( \sigma'^{-1} \) can be expressed using the same transpositions used to represent \( \sigma' \) but in opposite order so \( \sigma'^{-1} = \sigma' \)
3. \( \sigma'' = \sigma(\sigma')^{-1} \) \( (-1)^{\sigma'(k)} = (-1) \)

for fixed \( \sigma' \) as \( \sigma \) runs from all permutations \( \sigma \) does \( \sigma' \).

\[
\det AB = \sum_{\sigma''} a_{i' j'} \prod_{k=1}^{n-1} b_{\sigma''(k)} \sigma''(k) = \prod_{k=1}^{n-1} b_{1 \sigma''(k)} \sigma''(k) \]

\[
= \det A \cdot \det B.
\]
since \( \det AB = \det A \cdot \det B \) we have

\[
\begin{align*}
1 \quad & \det AB = \det A \det B = \det B \det A = \det BA \\
2 \quad & \det A^{-1} = \det I = 1 = \det A \cdot \det A^{-1}
\end{align*}
\]

This means

\[
\det (A^{-1}) = \frac{1}{\det (A)}
\]

\[
(2 \quad & A' = BAB^{-1} \\
\det A' = \det B \det A \frac{1}{\det B} = \det A
\]

change of basis

\[
|u\rangle = \sum |u_n\rangle c_n = \sum |w_n\rangle d_n
\]

assume both bases are orthonormal

\[
C_n = \langle v_n | v \rangle
\]

\[
\begin{align*}
C_n &= \langle v_n | v \rangle = \sum_{k} \langle v_n | w_k \rangle d_k \\
d_n &= \langle w_n | v \rangle = \sum_{k} \langle w_n | w_k \rangle C_k
\end{align*}
\]

similarly for operators.
\[ A_{\nu n} = \sum_{k} A_{\nu k} A_{k n}^{(v)} \]
\[ A_{\nu n} = \sum_{k} A_{\nu k} A_{k n}^{(w)} \]

To relate \( A_{\nu n} \) and \( A_{\nu n}^{(w)} \),

\[ A_{\nu w} = \sum_{k} A_{\nu k} A_{k w}^{(w)} = \sum_{n} A_{wn}^{(v)} \langle v_{m l w} \rangle A_{k n}^{(w)} \]

\[ A_{\nu e} \langle v_{e l w} \rangle = \sum_{n} A_{ek}^{(v)} A_{k n}^{(w)} \langle v_{e l w} \rangle \]

equating coefficients of the basis vectors \( \nu e \) gives

\[ \langle v_{m l w} \rangle A_{k n}^{(w)} = A_{m e}^{(v)} \langle v_{e l w} \rangle \quad (1) \]

Since

\[ c_{\nu} = \sum \langle v_{n l w} \rangle d_{k} = \sum \langle v_{n l w} \rangle \langle w_{k \nu} \rangle c_{\nu e} \]

mes means

\[ \sum_{k} \langle v_{n l w} \rangle \langle w_{k \nu} \rangle = \delta_{m n} \quad (2) \]

right multiply (1) by \( \langle w_{n l v} \rangle \) to get

\[ \sum_{k} A_{k n}^{(w)} \langle w_{n l v} \rangle A_{k w}^{(w)} = \]
\[ \sum_{e} A_{m e}^{(v)} \langle v_{e l w} X w_{n l v} \rangle = A_{m s}^{(v)} \]

\[ A_{m s}^{(v)} = \langle v_{m l w} \rangle A_{k n}^{(w)} \langle w_{n l v} \rangle \]
This has the claim

\[ A' = BAB'^{-1} \]

Taking determinants,

\[ \det A' = \det BAB'^{-1} = \det A \]

In this case the operator was the same but the matrix elements were in a different bases. This means \( \det A \) is independent of the choice of bases.