Lecture 2

Last time we introduced complex numbers

\[ P(z) = z^2 + 1 = (z - i)(z + i) \]
\[ i^2 = -1 \]

In general

\[ z = x + iy = \text{complex number} \]
\[ x, y = \text{real numbers} \]
\[ x = \text{real part of } z = \text{Re}(z) \]
\[ y = \text{imaginary part of } z = \text{Im}(z) \]

\[ |z| = \sqrt{x^2 + y^2} = \text{modulus of } z \]

\[ \phi: \cos\phi = \frac{x}{|z|}, \sin\phi = \frac{y}{|z|}, \text{argument of } z \]

\[ z = |z| e^{i\phi} \]

Complex plane
we defined addition, subtraction (like adding vectors in complex plane), multiplication and division

complex functions of complex arguments

\[ p_n(z) = c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n \]

complex polynomials. Later we will prove that \( p_n(z) \) always has \( n \) complex roots - this is called the fundamental theorem of algebra. Even though we introduced \( i \) to solve \( z^2 + 1 = 0 \); nothing additional is needed to factor any polynomial.

other functions

consider

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1} + \frac{x^2}{2 \cdot 1} + \frac{x^3}{3 \cdot 2 \cdot 1} + \ldots \]
For real numbers this series is absolutely convergent.

To see this consider the difference

\[ |e - \sum_{n=0}^{N} \frac{x^n}{n!}| = \]

using the series for \(e^x\) this becomes

\[ \sum_{n=N+1}^{\infty} \frac{x^n}{n!} \]

using the \(\Delta\) inequality

\[ \sum_{n=N+1}^{\infty} \frac{|x|^n}{n!} \]

Letting \(m = n-N-1\); \(n = m+N+1\)

\[ \sum_{m=0}^{\infty} \frac{|x|^{m+N+1}}{(m+N+1)!} = \frac{|x|^{N+1}}{(N+1)!} \left(1 + \frac{|x|}{(N+2)} + \frac{|x|^2}{(N+2)(N+3)} + \cdots\right) \]

\[ \leq \frac{|x|^{N+1}}{(N+1)!} \sum_{m=0}^{\infty} \left(\frac{|x|}{N+2}\right)^m = \frac{|x|^{N+1}}{(N+1)!} \frac{1}{1 - \frac{|x|}{N+2}} \]

provided \(N+2 > |x|\) for \(\frac{|x|}{N+2} < \frac{1}{2}\) this is bounded by

\[ \frac{|x|^{N+1}}{(N+1)!}, \quad 2 \]
which vanishes in the limit that \( n \to \infty \) for fixed \( x \). This means that the partial sums get closer and closer to \( e^x \).

Given this, we define the complex exponential by

\[ e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \]

Following what was done in the real case,

\[ |e^z - \sum_{n=0}^{N} \frac{z^n}{n!}| = \left| \sum_{n=N+1}^{\infty} \frac{z^n}{n!} \right| \]

Since \( |z| \leq 2 \), \( 1 \) is a metric we can use the \( \Delta \) inequality

\[ \leq \sum_{n=N+1}^{\infty} \frac{|z|^n}{n!} \]

Since \( |z| \) is real this converges for the same reason \( e^x \) converges.
properties of the exponential function

\[ e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \]

we can also define

\[ e^{-z} = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \]

adding and subtraction

\[ \frac{1}{2} e^z + \frac{1}{2} e^{-z} = \sum_{n=0}^{\infty} \frac{1 + (-1)^n}{2} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{z^{2n}}{2 \cdot 2n!} = \cosh z \]

\[ \frac{1}{2} e^z - \frac{1}{2} e^{-z} = \sum_{n=0}^{\infty} \frac{1 - (-1)^n}{2} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = \sinh z \]

This defines 2 new functions:

\[ \cosh z = \frac{1}{2} (e^z + e^{-z}) \]
\[ \sinh z = \frac{1}{2} (e^z - e^{-z}) \]
It follows from these definitions

\[ e^z = \cosh z + \sinh z \]

\[ e^{-z} = \cosh z - \sinh z \]

We can also replace \( z \) by \( iz \)
in all of these expressions

\[
\cosh(iz) = \sum_{n=0}^{\infty} \frac{(iz)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = \cos(z)
\]

\[
\sinh(iz) = \sum_{n=0}^{\infty} \frac{(iz)^{2n+1}}{(2n+1)!} = i \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = i \sin(z)
\]

This gives

\[ \sin z = -i \sinh(iz) \]
\[ \cos z = \cosh(iz) \]

This defines the sine and cosine of a complex argument.
we also have

\[ e^{iz} = \cosh iz + \sinh (iz) \]
\[ = \cos(z) + i \sin(z) \]

\[ e^{-iz} = \cosh (-iz) + \sinh (-iz) \]
\[ = \cos(z) - i \sin(z) \]

where we use the fact that \( \cosh z \) is an even function of \( z \) and \( \sinh z \) is an odd function of \( z \).

Note that the complex sine and cosine are not periodic functions but

\[ \cos(z) = \frac{1}{2} (e^{iz} + e^{-iz}) \]
\[ \sin(z) = \frac{1}{2i} (e^{iz} - e^{-iz}) \]
For homework - use the series representation to show

\[ e^{z_1} e^{z_2} = e^{z_1 + z_2} \]

using this result

\[
\cos^2 z + \sin^2 z =
\]

\[
\left( \frac{1}{2} \left( e^{iz} + e^{-iz} \right) \right)^2 + \left( \frac{1}{2i} \left( e^{iz} - e^{-iz} \right) \right)^2
\]

\[
\frac{1}{4} (e^{2iz} + e^{-2iz} + 2) - \frac{1}{4} (e^{-2iz} - 2 + e^{2iz}) =
\]

\[
\frac{2}{4} + \frac{2}{4} = 1
\]

which is identical to the result for real \( x \)

For homework I will ask you to prove

\[
\cos(z_1 + z_2) = \cos(z_1) \cos(z_2) - \sin(z_1) \sin(z_2) \]
\[
\sin(z_1 + z_2) = \sin(z_1) \cos(z_2) + \cos(z_1) \sin(z_2)
\]
Using this identity we can prove a number of useful results:

\[
\cos(x + iy) = \cos(x) \cos(iy) - \sin(x) \sin(iy)
\]
\[
= \cos(x) \cosh(y) - i \sinh(x) \sinh(y)
\]
\[
\sin(x + iy) = \sin(x) \cos(iy) + \cos(x) \sin(iy)
\]
\[
= \sin(x) \cosh(y) + i \cos(x) \sinh(y)
\]

\[
\cosh(x + iy) = \cos(ix - y) =
\]
\[
= \cos(ix) \cos(-y) - \sin(ix) \sin(-y)
\]
\[
= \cosh(x) \cos(y) + i \sinh(x) \sin(y)
\]
\[
\sinh(x + iy) = \sinh(i(y - ix)) = i \sinh(y - ix)
\]
\[
= i \sin(y) \cosh(-x) +
\]
\[
i(i) \cos(y) \sinh(-x)
\]
\[
= \cos(y) \sinh(y) + i \sin(y) \cosh(x)
\]

We can use these definitions to define a complex natural logarithm.

We require

\[e^{\ln z} = z\]
using this - if we assume $ln z = u + iv$
then
$$e^{u+iv} = z = |z|e^{i\phi}$$
$$= e^u e^{iv}$$

comparing these expressions
$$|z| = e^u \quad u = ln |z|$$
$$v = \phi + 2\pi n$$

This gives
$$ln z = ln |z| + i(\phi + 2\pi n)$$

In this case as $\phi$ increases the $ln z$ does not come back to its original value when $\phi$ increases by $2\pi$.

The complex natural log is an example of a multivalued function.
\[
\ln Z_1, Z_2 = \ln (1Z_1, 1e^{i\phi_1}, 1Z_2, 1e^{i\phi_2}) \\
= \ln (1Z_1, 1Z_2, e^{i(\phi_1 + \phi_2)}) \\
= \ln (1Z_1, 1Z_2, 1) + i(\phi_1 + \phi_2 + 2\pi n) \\
= (\ln 1Z_1 + i(\phi_1 + 2\pi n_1)) + (\ln 1Z_2 + i(\phi_2 + 2\pi n_2)) \\
= \ln Z_1 + \ln Z_2
\]

Note
\[
\ln Z_1, Z_2 = e^{\ln Z_1, Z_2} = e^{\ln Z_1} = e^{\ln Z_2} = e^{\ln Z_1} \\
\ln Z_1 = \ln e^{\ln Z_1} = \ln Z_2 \\
\therefore \ln Z_1 = Z_2 \ln Z_1
\]

These are standard properties of the real \(\ln\) - they carry over to the complex \(\ln\). The only new property is that \(\ln Z\) is a many-valued function.
The functions that we discussed were constructed from the exponential function.

In general we may not know how to define complex functions of a complex variable. In some cases they can be defined as limits of known functions.

**Def.** \( f_n(z) \) converges pointwise to \( f(z) \) if for every \( \epsilon > 0 \) there is an \( N(\epsilon, z) \) with the property that for every \( n > N(\epsilon, z) \):

\[
|f_n(z) - f(z)| < \epsilon
\]

The convergence is uniform if \( N(\epsilon, z) \) is independent of \( z \).

To test this it is necessary to have \( f(z) \) like we did for \( e^z \).
If we only know the \( s_n(z) \) we can construct \( f(z) \) provided \( s_n(z) \) is a Cauchy sequence.

\( s_n(z) \) is a Cauchy sequence at \( z \) if for every \( \varepsilon > 0 \) there is an \( N(z,\varepsilon) \) with the property that for \( m, n > N(z,\varepsilon) \)

\[
|s_n(z) - s_m(z)| < \varepsilon
\]

We note that convergent sequences are Cauchy sequences, while Cauchy sequences are convergent.

1. If \( s_n(z) \) converges to \( f(z) \) choose \( N(\frac{\varepsilon}{2}, z) \) so if \( n > N(\frac{\varepsilon}{2}, z) \) then

\[
|s_n(z) - f_m(z)| = |s_n(z) - f(z) + f(z) - s_m(z)| \\
\leq |s_n(z) - f(z)| + |s_m(z) - f(z)| \\
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]

This shows every convergent sequence is a Cauchy sequence.
If \( f_n(z) \) is a Cauchy sequence at \( z \), choose a subsequence \( \{ f_{n_k} \} \) satisfying \( N > n_k \)

\[
|f_n - f_{n_k}| < \frac{1}{2^k}
\]

Then define

\[
S(z) = f_{n_1}(z) + \sum_{m=0}^{\infty} (f_{n_m}(z) - f_{n_{m+1}}(z))
\]

we claim this series is convergent.

Choose \( N > n_k \)

\[
|S(z) - f_n(z)| = \left| f_{n_1}(z) - f_{n_k}(z) + \sum_{m=k}^{\infty} (f_{n_m}(z) - f_{n_{m+1}}(z)) \right|
\]

\[
\leq \frac{1}{2^k} + 2 \sum_{m=k}^{\infty} \frac{1}{2^m} = \frac{1}{2^k} \left( 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots \right)
\]

\[
= \frac{1}{2^k} \left( \frac{1}{1 - \frac{1}{2}} \right) = \frac{1}{2^{k+1}}
\]

This can be made as small as desired by choosing \( k \) sufficiently large.
These concepts generalize to more general classes of metric spaces.

The convergence is uniform if $N_k$ is independent of $z$.

Note that polynomial approximations to $e^x$ converge, but the convergence is not uniform. For a given $e$, more terms are needed if $x$ is large.

The purpose of this in complex analysis is that we need to have ways to define complex valued functions. We will discuss some other functions.

Now that we have defined complex functions, we can start doing calculus.

We define the complex derivative of a function $f(z)$ at $z = a$ as:

$$
\frac{df}{dz}(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}
$$

where we require that the limit exists and is unique.
The uniqueness is a strong constraint.

To see this, let $\Delta z = c e^{i\phi}$

$$\frac{df}{dz} = \lim_{c \to 0} \frac{f(z + c e^{i\phi}) - f(z)}{c e^{i\phi}}$$

for this to exist and be unique, it must exist and be independent of $\phi$.

To consider the implications,

$$\frac{df}{dz} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x + i\Delta y) - f(x + i\Delta y)}{\Delta x}$$

$$= \frac{i}{\Delta y} \frac{df}{d\Delta y}$$

Uniqueness requires

$$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y} = -i \frac{\partial \overline{f}}{\partial y}$$

If $f(z) = u(x,y) + i v(x,y)$, then

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = i(i) \frac{\partial v}{\partial y} = -i \frac{\partial u}{\partial y}$$
equating the real and imaginary parts
gives the following relations between
the real and imaginary parts of $f$

\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \]

These equations are called the
Cauchy-Riemann equations. They
are a necessary condition for
$f(z) = u(x) + iv(y)$ to have a
complex derivative.

Using these equations, we find

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial u}{\partial y^2} \]

\[ \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x^2} \]

This means

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u(x,y) = 0 \]

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) v(x,y) = 0 \]
This means that both the real and imaginary part of a function with a complex derivative separately satisfy Laplace's equation:

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f(x, y) = 0
\]

in 2 dimensions.

While satisfying Laplace's equations is necessary, it is not sufficient since they do not imply any relation between \( u \) and \( v \), whereas the Cauchy-Riemann equations require such a relation.

**Definition**: \( f(z) \) has a complex derivative at \( z \) then \( f(z) \) is called analytic at \( z \).

To understand the geometric meaning of the Cauchy-Riemann equations consider the curves in the \( xy \) plane defined parametrically by
\[ u(x,y) = c, \quad v = c \]
\[ v(x,y) = c \]

The gradients of these curves are:

\[ \nabla u = \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right), \quad \nabla v = \left( \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right) \]

we have

\[ \nabla u \cdot \nabla v = \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) = 0 \]

we see the gradients of the surfaces of constant \( u \) and constant \( v \) are perpendicular.

(Now on the surface of constant \( u \),
if \( x(s), y(s) \) is a path on a surface of constant \( u \),
\[ \frac{\partial u}{\partial x} \frac{dx}{ds} + \frac{\partial u}{\partial y} \frac{dy}{ds} = 0 \]
\[ \left( \frac{dx}{ds}, \frac{dy}{ds} \right) \text{ is tangent to } u = \text{const} \)
examples of analytic functions

1. Polynomials

\[ P(z) = \sum_{n=0}^{\infty} c_n z^n \]

\[ \frac{d}{dz} (z^n) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^n - z^n}{\Delta z} \]

\[ = \lim_{\Delta z \to 0} \sum_{m=0}^{n} \frac{n!}{m! (n-m)!} z^m \Delta z^{n-m} = z^n \Delta z \]

when \( m = n \)

\[ \frac{n!}{n!} \Delta z \]

which means that the sum \( n \to n-1 \)

\[ \lim_{\Delta z \to 0} \sum_{m=0}^{n-1} \frac{n!}{m! (n-m)!} z^m \Delta z^{n-m-1} \]

the only non vanishing term is the one when \( n-m-1 = m = n-1 \)

\[ = \frac{n!}{(n-1)! 1!} \Delta z \]

\[ = n z^{n-1} \Delta z \to 0 \]
Since the derivative is linear
\[ \frac{dP_n}{dz} = \sum_{n=0}^N c_n \frac{dz^n}{dz} = \sum_{n=1}^N n c_n z^{n-1} \]

to understand the meaning of the Cauchy-Riemann equations, let \( f(xy) = u(xy) + iv(xy) \) and assume \( f(xy) \) has a convergent Taylor series
\[ f(xy) = \sum_{nm=0}^\infty \frac{x^n y^m}{n! m!} \frac{\partial^m}{\partial x^n \partial y^m} (u + iv) \bigg|_{x=y=0} \]
The Cauchy-Riemann equations imply
\[ \frac{\partial}{\partial y} (u + iv) = -\frac{\partial v}{\partial x} + i \frac{\partial u}{\partial x} = i \frac{\partial}{\partial x} (u + iv) \]
using this to change the \( y \) derivative to \( x \) derivative gives
\[ f(xy) = \sum_{nm=0}^\infty \frac{x^n y^m}{n! m!} \frac{\partial}{\partial x^n} \frac{\partial}{\partial y^m} (u + iv) \bigg|_{x=y=0} \]
\[ = \sum_{nm=0}^\infty \frac{x^n (iy)^m}{n! m!} \frac{\partial^{n+m}}{\partial x^n \partial (iy)^m} (u + iv) \bigg|_{x=y=0} \]

Let \( k = m+n \)
\[ = \sum_{k=0}^\infty \frac{1}{k!} \frac{k!}{m! (k-m)!} x^{k-m} (iy)^m \frac{\partial^k}{\partial x^k} (u + iv) \]
\[
= \sum_{k=0}^{\infty} \frac{1}{k!} (x+iy)^k \frac{\partial^k}{\partial x^k} (u+iv)\bigg|_0
\]

The implication is that if \( f(z) \) satisfies the Cauchy-Riemann equations

\[ f(xy) = f(z \bar{z}^x) \]

only depends on \( z \), not \( \bar{z} \).

It is also clear in this form that this function will have a complex derivative

\[
\frac{df}{dz} = \frac{\partial f}{\partial x} \frac{dx}{dz} + i \frac{\partial f}{\partial y} \frac{dy}{dz}
= \frac{\partial f}{\partial x} \frac{i}{2} + \frac{\partial f}{\partial y} \frac{i}{2} = 0
\]

\[ x = \frac{1}{2}(z + \bar{z}) \]
\[ y = \frac{1}{2}(z - \bar{z}) \]

which is the uniqueness requirement that gave us the Cauchy-Riemann equations.