Lecture 3

Complex derivatives

\[
\frac{d f(z)}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}
\]

provided the limit exists and is unique

The uniqueness arises because

\[
\Delta z = |\Delta z| e^{i\phi}
\]

it is important that the result is independent of \( \phi \) as \( |\Delta z| \to 0 \)

Implications assume that \( \frac{df}{dz} \) exists. Then we can compute the derivative using

\[
\Delta z = \Delta x
\]

or

\[
\Delta z = i \Delta y
\]

The result should be independent of this choice
\[
\frac{df}{dz} = \frac{d}{dx} f(x+iy) = \frac{df}{dx} \]
\[
= \frac{d}{d(iy)} f(x+iy) = \frac{i}{i} \frac{df}{dy} = -i \frac{df}{dy}
\]

Equating these partial derivatives give:
\[
\frac{df}{dz} = \frac{df}{dx} = -i \frac{df}{dy}
\]

If we consider the real and imaginary parts of \(f\)
\[
\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \left( \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right)
\]

Comparing the real and imaginary parts gives the pair of equations:
\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}
\]

These equations are called the Cauchy-Riemann equations.

They show that \(u\) and \(v\) are related if \(f\) has a complex derivative.
Note
\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial y} \left( -\frac{\partial u}{\partial y} \right) = -\frac{\partial^2 u}{\partial y^2},
\]
\[
\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x} \left( -\frac{\partial v}{\partial y} \right) = -\frac{\partial}{\partial y} \left( \frac{\partial v}{\partial x} \right) = -\frac{\partial^2 v}{\partial y^2}.
\]

This means
\[
\begin{align*}
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u(x,y) &= 0 \\
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) v(x,y) &= 0
\end{align*}
\]

These equations are called Laplace's equation in 2 dimensions. The real and imaginary parts separately satisfy Laplace's equation, but this is not sufficient for the existence of a derivative since they do not relate \(u\) and \(v\).

Consider curves in the complex \(xy\) plane defined by
\[
\begin{align*}
u(x,y) &= \text{constant} \\
v(x,y) &= \text{constant}
\end{align*}
\]
If $x(u, v)$ satisfies $u(x(u), y(u)) = c$
then
\[
\frac{d}{dx} (\xi) = 0 = \frac{\partial u}{\partial x} \frac{dx}{d\xi} + \frac{\partial u}{\partial y} \frac{dy}{d\xi}
\]
\[
(\frac{dx}{d\xi}, \frac{dy}{d\xi}) \text{ is the tangent vector along this curve}
\]
\[
(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = \vec{\nabla}u(xy) \text{ is the gradient of } u
\]

The above equation implies that
\[
\vec{\nabla}u \cdot \vec{t} = 0
\]
are the gradient is $\perp$ to the curve $x(u, v)$, so $\vec{\nabla}u$ is normal to the curve $u(xy) = c$

Similarly, $\vec{\nabla}v$ is normal to the curve $v(xy) = c$

The Cauchy–Riemann equations imply

\[
\vec{\nabla}u \cdot \vec{\nabla}v = \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} = 0
\]
\[
= \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \bigg|_{xy} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \bigg|_{xy} = 0
\]

This means that the normals to the surfaces on constant $u$ and constant $v$ are $\perp$. 


Examples

Polynomials

Consider $f(z) = z^n$

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^n - (z)^n}{\Delta z} =$$

$$\frac{1}{\Delta z} \left( \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z^k \Delta z^{n-k} - z^n \right) =$$

$$\frac{1}{\Delta z} \left( z^n + \frac{n!}{(n-1)!} z^{n-1} \Delta z + \frac{n!}{(n-2)!} z^{n-2} \Delta z^2 + \cdots - z^n \right) =$$

$$n z^{n-1} + \frac{n(n-1)}{2!} z^{n-2} \Delta z + \frac{n(n-1)(n-2)}{3!} z^{n-3} \Delta z^2 + \cdots$$

as $\Delta z \to 0$ what remains is $n z^{n-1}$.

In this case the result has no dependence - it is unique so

$$\frac{d}{dz}(z^n) = n z^{n-1}$$
since the derivative is linear

\[
\frac{d}{dz} \left( f_1(z) + f_2(z) \right) = \lim_{\Delta z \to 0} \frac{f_1(z+\Delta z) + f_2(z+\Delta z) - f_1(z) - f_2(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{f_1(z+\Delta z) - f_1(z)}{\Delta z} + \lim_{\Delta z \to 0} \frac{f_2(z+\Delta z) - f_2(z)}{\Delta z}
\]

\[
= \frac{df_1}{dz} + \frac{df_2}{dz}
\]

provided both derivatives exist.

It follows that

\[
\frac{d}{dz} \left( \sum_{n=0}^{N} c_n z^n \right) = \sum_{n=0}^{N} \frac{d}{dz} \left( z^n \right) = \sum_{n=1}^{N} n c_n z^{n-1}
\]

which shows polynomials are differentiable, and they generalized the result for real polynomials.

What does it mean to be differentiable? Let \( f(z) = f(x, y) \) be an arbitrary function of \( x, y \).
Since \( z = x + iy \), \( z^* = x - iy \) in general.

\[
\begin{align*}
\frac{\partial f}{\partial z} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial z} \\
\frac{\partial f}{\partial z^*} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial z^*} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial z^*}
\end{align*}
\]

Note

\[
\begin{align*}
x &= \frac{1}{2} (z + z^*) \\
y &= \frac{i}{2} (z^* - z)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial x}{\partial z} &= \frac{1}{2} \\
\frac{\partial y}{\partial z} &= \frac{i}{2}
\end{align*}
\]

In a function with a complex derivative

\[
\frac{\partial f}{\partial z} = -i \frac{\partial f}{\partial y}
\]

Using this in the equation above means

\[
\frac{\partial f}{\partial z} = 0
\]

This means that in order to have a complex derivative \( f(x, y) \) considered as a function of \( z, z^* \) must be independent at \( z \). This is what the Cauchy-Riemann equations imply.
Because \( e^z \) has a convergent power series, \( e^z \) has a complex derivative for all \( z \) in the complex plane.

This applies to \( \sin h z \), \( \cosh z \), \( \sin z \), and \( \cos z \).

A function \( f(z) \) is **analytic** at \( z \) if

1. \( f(z) \) has a complex derivative at \( z \)
2. There is a neighborhood of \( z \) where \( f(z) \) is single valued.

It is also possible to integrate complex functions. The most common cases are integrals along a curve in the complex plane,

\[
\gamma(\lambda) = \gamma_x(\lambda) + i \gamma_y(\lambda)
\]

\( \gamma(a) \)

\( \gamma(b) \)

Here \( \lambda \) is a real parameter that varies between \( a \) and \( b \), which could be limit or infinite.
\[ \int_{z_i}^{z_{i+1}} f(z) \, dz \]

This is defined like an ordinary integral:

\[ \int_{z_i}^{z_{i+1}} f(z) \, dz = \lim_{\Delta z_i \to 0} \sum_{i=1}^{2} f(z_{1i}) \Delta z_i \]

Evaluated at a point along the curve between \( z_i \) and \( z_{i+1} \):

\[ \lim_{\Delta z_i \to 0} \left( u(z_i) \Delta x_i + i v(z_i) \Delta y_i + i u(z_{i+1}) \Delta x_i + u(z_{i+1}) \Delta y_i \right) \]

This uses ordinary calculus:

\[ \int_{a}^{b} \left( u'(\lambda) + iv'(\lambda) \right) d\lambda \]

\[ = \int_{a}^{b} u'(\lambda) \, d\lambda + i \int_{a}^{b} v'(\lambda) \, d\lambda \]

\[ = \int_{a}^{b} \left( \frac{dy}{d\lambda} + i \frac{dx}{d\lambda} \right) \, d\lambda \]

\[ + \int_{a}^{b} \left( u'(\lambda) \frac{dy}{d\lambda} - v'(\lambda) \frac{dx}{d\lambda} \right) \, d\lambda \]

Examples of path:

\[ \chi(z) = R e^{2\pi i \lambda} = R(\cos(2\pi \lambda) + i \sin(2\pi \lambda)) \]

\( \lambda \in [0,1] \)

Circle of radius \( R \) centered about the origin.
Example 2

\[ y(\lambda) = \begin{cases} 
10\lambda & 0 \leq \lambda \leq \frac{1}{2} \\
5 + 8i(\lambda - \frac{1}{2}) & \frac{1}{2} \leq \lambda \leq 1 
\end{cases} \]

\[ f(z) = z^2 \]

\[ \int_y z^2 dz = \int_0^1 (y_x + i y_y)^2 \left( \frac{dy_x}{d\lambda} + i \frac{dy_y}{d\lambda} \right) = \]

\[ \int_0^1 \left( y_x^2 - y_y^2 + 2iy_x y_y \right) \left( \frac{dy_x}{d\lambda} + i \frac{dy_y}{d\lambda} \right) = \]

\[ \int_0^1 \left( (y_x^2 - y_y^2) \frac{dy_y}{d\lambda} - 2y_x y_y \frac{dy_x}{d\lambda} \right) d\lambda + \]

\[ i \int_0^1 \left( 2y_x y_y \frac{dy_y}{d\lambda} + (y_x^2 - y_y^2) \frac{dy_x}{d\lambda} \right) d\lambda \]

On \( 0 \to \frac{1}{2} \), \( y_x = 10 \times 0 = 0 \), \( \frac{dy_x}{d\lambda} = 10 \), \( \frac{dy_y}{d\lambda} = 0 \)

On \( \frac{1}{2} \to 1 \), \( y_x = 10 \), \( y_y = 8(\lambda - \frac{1}{2}) \), \( \frac{dy_x}{d\lambda} = 0 \), \( \frac{dy_y}{d\lambda} = 8 \)

To evaluate the integral let

\[ S_0^1 d\lambda = S_0^{\frac{1}{2}} d\lambda + S_{\frac{1}{2}}^1 d\lambda \]
putting everything together gives

\[ \int f(z) \, dz = \]

\[ 8 \int_0^1 ((10 \lambda)^2 \, 10 \, d\lambda + \]

\[ \int_{\frac{1}{2}}^1 \left[ -10 \cdot 8(\lambda-\frac{1}{2}) \cdot 8 + i \left( 10^2 - 8^2(\lambda-\frac{1}{2})^2 \right) \right] \, d\lambda = \]

\[ 10^3 \int_0^{\frac{1}{2}} x^2 \, d\lambda + (320 - i \cdot 8^3 \cdot \frac{1}{4}) \int_{\frac{1}{2}}^1 \]

\[ + (-640 - i \cdot 8^3) \int_{\frac{1}{2}}^1 \lambda \, d\lambda + \]

\[ + i \cdot 8^3 \int_{\frac{1}{2}}^1 \lambda^2 \, d\lambda \]

These can be added together, the result is a complex number.

Example 3

Let \( y(\lambda) = R(\cos(2\pi \lambda)) + i \sin(2\pi \lambda) \)

\( f(z) = z \)

In this case the path is a closed circle in the counter clockwise direction.
In this case

\[
\begin{align*}
\gamma_x &= R \cos(2\pi \rho) \\
\gamma_y &= R \sin(2\pi \rho) \\
\frac{dx}{d\rho} &= -R \cdot 2\pi \sin(2\pi \rho) \\
\frac{dy}{d\rho} &= R \cdot 2\pi \cos(2\pi \rho)
\end{align*}
\]

\[
\int_y f(z) \, dz =
\]

\[
\int_0^1 \left( R \cos(2\pi \rho) + i R \sin(2\pi \rho) \right) \left( -2\pi R \sin(2\pi \rho) + i 2\pi R \cos(2\pi \rho) \right)
\]

\[
\int_0^1 -2\pi R^2 \left( \cos(2\pi \rho) \sin(2\pi \rho) + \cos(2\pi \rho) \sin(2\pi \rho) \right)
\]

\[
i \left( -\sin^2(2\pi \rho) + \cos^2(2\pi \rho) \right)
\]

Recall from

\[
2 \sin x \cos x = \sin 2x
\]
\[
\cos^2 x - \sin^2 x = \cos 2x
\]

\[
\int_y f(z) \, dz =
\]

\[
-2\pi R^2 \int_0^1 \sin(4\pi \rho) + i 2\pi R^2 \int_0^1 \cos(4\pi \rho)
\]

\[
\frac{2\pi R^2}{4\pi} \cos(4\pi \rho) \bigg|_0^1 + \frac{2\pi R^2}{4\pi} \sin(4\pi \rho) \bigg|_0^1 = 0 + 0 i
\]
when the integral is over a closed path we use the symbol
\[ \oint_c f(z) \, dz \]

in this example
\[ \oint_c z \, dz = 0 \]

example 4 - consider the same path where \( f(z) = \frac{1}{z} \) instead of \( z \)
\[ \oint_c f(z) \, dz = \oint_0^1 \frac{dz}{z} \]

in this case it is useful to use
\[ z = R e^{2 \pi i \lambda}, \quad \frac{1}{z} = \frac{1}{R} e^{-2 \pi i \lambda} \]
\[ dz = 2 \pi i R e^{2 \pi i \lambda} \, d \lambda \]

then
\[ \oint_c \frac{dz}{z} = \oint_0^1 \frac{1}{R} e^{-2 \pi i \lambda} \times R 2 \pi i e^{2 \pi i \lambda} \, d \lambda \]
\[ = 2 \pi i \]
The integrals in examples 3 and 4 will be important later.

Example 5

\[ \oint_C \ln z \, dz \]

where \( C \) is a closed circle of radius \( R \) about the origin.

\[ C = R e^{2\pi i a} \]

\[ \ln C = \ln R + i (2\pi a + 2\pi k) \]

\[ d\lambda = 2\pi i R e^{2\pi i a} \, d\lambda \]

The integral is

\[ \oint_C (\ln R + i (2\pi a + 2\pi k)) \frac{2\pi i R e^{2\pi i a}}{2\pi i} \, d\lambda = \]

\[ (\ln R + 2\pi i n) 2\pi R i \int_0^1 e^{2\pi i a} \, d\lambda \]

\[ = 4\pi^2 R \int_0^1 x e^{2\pi i a} \, dx \]

The first term vanishes because

\[ \int_0^1 e^{2\pi i a} \, dx = \frac{1}{2\pi i} (e^{2\pi i} - 1) = 0 \]

for the second integral

\[ \int_0^1 x e^{ax} \, dx = \frac{d}{da} \int_0^1 e^{ax} \, dx = \frac{d}{da} \frac{1}{a} (e^{ax} - e^a) \]
\[ -\frac{1}{2\pi i} (e^{\alpha b} - e^{\alpha a}) + \frac{1}{\alpha} (e^{\alpha b} - e^{\alpha a}) \]

In the case of the second integral \( \alpha = 2\pi i \)
\( \alpha b = 2\pi i \quad \alpha a = 0 \quad e^{\alpha b} - e^{\alpha a} = 1 - 1 = 0 \)

\[ -4\pi^3 R \int_0^1 \lambda e^{2\pi i \lambda} d\lambda = -4\pi^3 R \frac{1}{2\pi i} (1 - e^{-2\pi i \lambda}) = 2\pi R i \]

We note that even though the curve returns to the same point, we get a contribution because the log is multiply valued - instead the phase increases by the arc length.

An important result that we will use is Darboux's theorem,

\[ \left| \int_{\gamma} f(z) dz \right| \leq \max_{z \in \gamma} |f(z)| \cdot L \]

length of curve

To show this
\[ \left| \int_\chi f(z) \, dz \right| = \left| \int_a^b f(y(x)) \frac{dy}{dx} \, dx \right| \leq \int_a^b |f(y(x))| |\frac{dy}{dx}| \, dx \leq \max_{\lambda \in \mathbb{C}, \eta} |f(y(\lambda))| \left\{ \int_a^b \left[ \left( \frac{dy}{dx} \right)^2 + \left( \frac{dp}{dx} \right)^2 \right]^{\frac{1}{2}} \, dx \right\}. \]

\[ \max_{\lambda \in \mathbb{C}, \eta} |f(y(\lambda))| \int_a^b \frac{dx}{dx} \, dx = \max_{\lambda \in \mathbb{C}, \eta} |f(y(\lambda))|. \]

**Definitions**

1. A complex valued function of a complex variable, \( f(z) \), is analytic at \( z \) if:

a) There is a neighborhood of \( z \) where \( f(z) \) is single valued.

b) \( f(z) \) has a complex derivative at \( z \).
(2) The domain of analyticity is the set of points in the complex plane where \( f(z) \) is analytic.

(3) A complex valued function of a complex variable is entire if it is analytic everywhere in the complex plane.

(4) A point where \( f(z) \) is not analytic is called a singular point of \( f(z) \).

(5) A singular point is isolated if there is a neighborhood of that point where \( f(z) \) is analytic (except at \( z \)).
examples

\[ f(z) = \frac{1}{z} \left( \frac{1}{z - z_0} \right) \]

has isolated singular points
at \( z = 0, z_0 \).

\[ f(z) = \ln z \] is not analytic
at \( z = 0 \) (no neighborhood where
\( \ln z \) is single valued).

Conformal Mapping

For ordinary functions
we are interested in the question

\[ y = g(x) \]

when can \( t \) be invented
to construct \( x = f(y) \)?

This problem arises in
classical mechanics - given
a Lagrangian \( L(q, \dot{q}) \),
we define

\[ p = \frac{\partial L}{\partial \dot{q}} (q, \dot{q}) \]
The Hamiltonian is given by the Legendre transform

\[ H(p,q) = p \dot{q} - L \]

to compute \( H \) we need to be able to invert

\[ p = \frac{\partial L}{\partial \dot{q}} (\dot{q},q) \rightarrow \]

\[ \dot{q} = f(p,q) \]

In mechanics the condition to be able to do this is

\[ \frac{\partial^2 L}{\partial q^2} = -\frac{\partial p}{\partial q} > 0 \]

geometrically

\[ p \dot{q} \quad \text{(line with slope \( p \))} \]

\[ \frac{\partial^2 L}{\partial q^2} > 0 \]

intersects convex curve once

(requires a kinetic energy that increases with velocity)
In the complex case
\[ z' = g(z) \quad z = f(z') ? \]
given \( g(z) \) when can be calculate \( f(z) \)
assume
\[ z_0' = g(z_0) \]
define
\[ R(z, z_0) = z' - z_0' - \frac{dg}{dz}(z)(z - z_0) \]
\[ = g(z') - g(z) - \frac{dg}{dz}(z)(z - z_0) \]
the existence of a complex derivative at \( z = z_0 \) means that
\[ g(z) = g(z) + \frac{dg}{dz}(z_0)(z - z_0) + R(z, z_0) \]
\[ (z - z_0) = \left( \frac{dg}{dz}(z_0) \right)^{-1} \]
\[ (g(z) - g(z) - R(z, z_0)) \]
\[ \left( \frac{dg}{dz}(z_0) \right)^{-1} (z' - z_0' - R(z, z_0)) \]
with
\[ \frac{R(z, z)}{z - z_0} \to 0 \text{ as } z \to z_0 \]
we assume \( \frac{dq}{dz}(z_0) \neq 0 \) so \( \left( \frac{dq}{dz}(z_0) \right)^{-1} \)

makes sense.

Note that

\[
Z = z_0 + \left( \frac{dq}{dz}(z) \right)^{-1} \left( \frac{dz'}{dz} - R(z, z) \right)
\]

we note that \( z \) appears on both sides of this equation, but we
know \( R \) is small so we try to solve this by iteration

\[
Z_1 = z_0 + \left( \frac{dq}{dz}(z_0) \right)^{-1} \left( \frac{dz'}{dz} - R(z_0, z_0) \right)
\]

\[
Z_2 = z_0 + \left( \frac{dq}{dz}(z_2) \right)^{-1} \left( \frac{dz'}{dz} - R(z_1, z_2) \right)
\]

\[
\vdots
\]

It can be shown that this iteration converges to an analytical
function.