Lecture 5

Darboux's Theorem

\[ \left| \int_\gamma f(z) \, dz \right| \leq \max_{z \in \gamma} |f(z)| \cdot L \]

where \( L \) is the length of the path \( \gamma \) in the complex plane

\[ \left| \int_\gamma f(z) \, dz \right| = \left| \int_0^b f'(x(t)) \, \frac{dx}{dt} \, dt \right| \leq \]

\[ \int_0^b |f'(x(t))| \, |\frac{dx}{dt}| \, dt \leq \max_{t \in [a,b]} |f'(x(t))| \cdot \int_0^b \frac{dx}{dt} \, dt \]

\[ \max_{t \in [a,b]} |f'(x(t))| \int_0^b \sqrt{\left( \frac{dx}{dx} \right)^2 + \left( \frac{dy}{dx} \right)^2} \, dt \]

but the last integral is the length of the path

\[ ds = \sqrt{dx^2 + dy^2} = \sqrt{(\frac{dx}{dx})^2 + (\frac{dy}{dx})^2} \, dt \]

This proves Darboux's Theorem — this will be used later.
Lecture 5

Assume \( f(z) \) is analytic at \( z = z_0 \) and satisfies \( \frac{df}{dz}(z_0) \neq 0 \). Then the mapping

\[
x' = x' + iy'
\]

defined by

\[
\begin{align*}
z' &= x' + iy' = f(z) = f(x + iy) \\
x' &= u(x, y) \\
y' &= v(x, y)
\end{align*}
\]

is called a **conformal mapping**

Theorem: Conformal mappings preserve the angle between curves

Proof: Let

\[
\begin{align*}
\gamma_1(\lambda) &= z_0 + r_1(\lambda) e^{i\phi_1(\lambda)} \\
\gamma_2(\lambda) &= z_0 + r_2(\lambda) e^{i\phi_2(\lambda)}
\end{align*}
\]

define the transformed curves

\[
\begin{align*}
\gamma_1'(\lambda) &= f(z_0) + r_1(\lambda) e^{i\phi_1(\lambda)} = f(\gamma_1) \\
\gamma_2'(\lambda) &= f(z_0) + r_2(\lambda) e^{i\phi_2(\lambda)} = f(\gamma_2)
\end{align*}
\]
Because $f(z)$ is analytic at $z = z_0$, we can compute the complex derivative at $z = z_0$ along any path - the resulting complex derivative is independent of the choice of path.

\[
\frac{df}{dz} = \lim_{\lambda \to 0} \frac{f(\gamma_1(\lambda)) - f(\gamma_1(0))}{\gamma_1(\lambda) - \gamma_1(0)} = \frac{f(z_0) + r_1(\lambda) e^{i\phi_1'(\lambda)} - f(z_0)}{z_0 + r_1(\lambda) e^{i\phi_1}(\lambda) - z_0} = \frac{r_1(0)}{r_1(0) e^{i\phi_1(0)}} \cdot \frac{r_1(0) e^{i\phi_1'(0)} - r_1(0) e^{i\phi_1(0)}}{r_1(0) e^{i\phi_1(0)}}
\]

The result has to be the same if we move along $\gamma_2(\lambda)$

\[
\frac{df}{dz} = \frac{r_2(0) e^{i(\phi_2'(\lambda) - \phi_2(\lambda))}}{r_2(0) e^{i\phi_2(0)}}
\]
Comparing these, we get
\[
\frac{r_1'(0)}{r_1(0)} = \frac{i(\phi_1'(0) - \phi_1(0))}{i(\phi_2'(0) - \phi_2(0))} = \frac{r_2'(0)}{r_2(0)}
\]
for these to be equal, the modulus and arguments must be the same
\[
\frac{r_1'(0)}{r_1(0)} = \frac{r_2'(0)}{r_2(0)}
\]
\[
\phi_1'(0) - \phi_1(0) = \phi_2'(0) - \phi_2(0) + 2\pi n
\]
We can write the second equation as
\[
\phi_2'(0) - \phi_1(0) = \phi_2'(0) - \phi_1(0) + 2\pi n
\]
In the plane, the \(2\pi n\) is irrelevant, which means that the difference in angles between \(\phi_1\) and \(\phi_2\) at \(z = z_0\) is the same.
\[
\phi_2(0) - \phi_1(0) = \phi_2'(0) - \phi_1'(0)
\]
This shows that \( xy \rightarrow x'y' \) is a transformation in the plane that preserves angles.

Note that when \( \frac{df}{dz}(z) \neq 0 \), then \( z' = f(z) \) an on an inverse in a neighborhood of \( z' = f(z) \).

To show this we write

\[
z' = f(z) + \frac{df}{dz}(z) (z-z_0) + R(z,z_0)
\]

where \( R(z,z_0) \) is the remainder.

\( R(z,z_0) \) is analytic since \( f(z) + \frac{df}{dz}(z) (z-z_0) \) is analytic and in addition since

\[
\frac{f(z)-f(z_0)}{z-z_0} = \frac{df}{dz}(z) + \frac{R(z,z_0)}{z-z_0}
\]

The existence of the derivative means \( R(z,z_0)/(z-z_0) \) can be made as small as desired by making \( z-z_0 \) small.
consider the equation
\[(z - z_0) = \left(\frac{df}{dz}(z_0)\right)^{-1} (z' - z_0' - R(z, z'))\]
by choosing \(z' - z'_0\) sufficiently small it is possible to make \(|R(z, z')| \ll (z' - z'_0)\)

we can define a sequence of successive approximations:
\[
(z_1 - z_0) = \left(\frac{df}{dz}(z_0)\right)^{-1} (z' - z'_0)
\]
\[
(z_2 - z_0) = \left(\frac{df}{dz}(z_1)\right)^{-1} (z_1' - z_1' - R(z_1, z_1'))
\]
\[
(z_3 - z_0) = \left(\frac{df}{dz}(z_2)\right)^{-1} (z_2' - z_2' - R(z_2, z_2'))
\]

in general
\[
(z_n - z_0) = \left(\frac{df}{dz}(z_{n-1})\right)^{-1} (z_n' - z_n' - R(z_n, z_n'))
\]

It can be shown that this sequence is a Cauchy sequence that converges to a function
\[
z = g(z') = \lim_{n \to \infty} (z_n + \left(\frac{df}{dz}(z_n')\right)^{-1} (z' - z'_n - R(z_n, z_n'))
\]

where
\[
\frac{dg(z)}{dz'}(z') = \left(\frac{df}{dz}(z_0)\right)^{-1}
\]
To complete the proof it is necessary to prove that the sequence is a Cauchy sequence. This is a consequence of the existence of the complex derivative.

The inverse \( g(z') = z \) is only a local inverse. It is only an inverse in \( z' \) near \( g(z) \).

Remark: This result is more general in classical mechanics a Lagrangian \( L(\dot{q}, q) = T(\dot{q}) - V(q) \) where \( T(\dot{q}) \) represents the kinetic energy as a function of coordinates and velocities.

The generalized momentum is defined by

\[
p = \frac{\partial L}{\partial \dot{q}}(\dot{q}, q)
\]

The Hamiltonian is a function of \( p \) and \( q \) defined by

\[
H(p, q) = p \dot{q} - L(q, \dot{q})
\]

\[
\frac{\partial H}{\partial \dot{q}} = p - \frac{\partial L}{\partial \dot{q}} = p - p = 0
\]
To find $\dot{q}$, we must be able to invent $P = \frac{\partial L}{\partial \dot{q}}$.

To find $\dot{q}(Pq)$, a sufficient condition to be able to do this is

$$\frac{\partial P}{\partial q} = \frac{\partial^2 L}{\partial q^2} > 0$$

Here we see $P$ is the slope of the line $\frac{\partial^2 L}{\partial q^2}$ means that the slope of $\frac{\partial L}{\partial q}$ keeps increasing.

From the picture you can see there is a unique intersection.

Application of conformal mapping.

\[ T = 0 \]

\[ \frac{\partial T}{\partial y} = 0 \]

$T = 1$
The thermal energy of the quoted plane is

\[ \int dE = \int L_c T(x,y) \, dx \, dy = E \]

where \( c \) is the heat capacity (we assume that it is a constant independent of temperature)

The only way the total energy of the plane can change if heat is transported across the boundary

\[ \frac{dE}{dt} = \oint \vec{J} \cdot d\vec{s} \]

where \( \vec{J} \) is the heat current - we assume that it is proportional to the thermal gradient, but in the opposite direction (i.e., heat travels from hot to cold)

\[ \vec{J} = -\kappa \nabla T \]

\[ \frac{d}{dt} \int c T(x,y) \, dx \, dy = -\kappa \oint \nabla T \cdot d\vec{s} \]
using the divergence theorem in 2 dimensions
\[ \oint \mathbf{V} \cdot d\mathbf{A} = \int \nabla \cdot \mathbf{V} \, dV \]
comparing both sides
\[ \frac{d}{dt} \int \mathbf{c} \cdot d\mathbf{A} = -\kappa \int \nabla^2 T \, dA \]
when the temperature reaches a steady state then \( \frac{dT}{dt} = 0 \)
which gives
\[ -\kappa \int \nabla^2 T \, dA = 0 \]
this leads to the equation
\[ \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \]
this in steady state the temperature is a solution of Laplace's equation
the solution depends on the boundary conditions - physical situation
we know that the solution satisfying the boundary conditions is a function \( u(x,y) \).

Let \( Z = x + iy \), \( W = q + ip \),

consider the change of variables

\[
Z = \sin \omega \\
= \sin (q + ip) \\
= \sin q \cosh p + i \sinh p \cos q \\
= x + iy
\]

Comparing these we get the change of variables

\[
X = \sin q \cosh p \\
Y = \sinh p \cos q
\]

We note

\[
u(x,y) = u(x(q,p), y(q,p)) = T(q,p)
\]

The boundary \( x = 0 \) \( y > 0 \) \( \Rightarrow q = 0 \) \( p > 0 \),

the boundary \( y = 0 \) \( 0 < x < 1 \) \( p = 0 \) \( 0 \leq q \leq \frac{\pi}{2} \),

the boundary \( y = 0 \) \( x > 1 \) \( q = \frac{\pi}{2} \) \( p > 0 \).

Note that

\[
T(\omega) = \begin{cases} 
\frac{1}{2} & \frac{\pi}{2} \\
q & 0 \\
0 & \omega = 0
\end{cases}
\]

\[
\frac{dT}{dp} = 0 \\
T(\omega) = 1 \\
q = \frac{\pi}{2}
\]
This is $\frac{2}{k} x$ real part of $\phi$ which trivially is a solution of Laplace equation with the transformed boundary conditions.

It follows that

$$T(x,y) = \frac{2}{k} \varphi(x,y)$$

is a solution satisfying the boundary conditions in $xy$.

The problem is to invent and find $\varphi(x,y)$.

\[
\begin{align*}
  x &= \sin \varphi \cosh \rho \\
  y &= \sinh \varphi \cos \vartheta \\
  \frac{x^2}{(\sin \varphi)^2} - \frac{y^2}{(\cos \varphi)^2} &= \cosh^2 \rho - \sinh^2 \rho = 1 \\
  \frac{x^2}{\sin^2 \vartheta} - \frac{y^2}{1 - \sin^2 \vartheta} &= 1 \\
  x^2 (1 - \sin^2 \varphi) - y^2 \sin \varphi &= \sin \varphi - \sin^3 \varphi \\
  \sin^4 \varphi - \sin^2 \varphi (1 + x^2 + y^2) + x^2 &= 0 \\
  \sin^2 \varphi &= \frac{(1 + x^2 + y^2) \pm \sqrt{(1 + x^2 + y^2)^2 - 4x^2}}{2}
\end{align*}
\]
There are 2 solutions. The correct solution should vanish when $x=y=0$.

This means

$$\sin^2 q = \frac{1+x^2+y^2 - \sqrt{(1+x^2+y^2)^2 - 4x^2}}{2}$$

$$T = \frac{2\pi}{\sin^{-1}\left(\frac{\sqrt{1+x^2+y^2 - \sqrt{(1+x^2+y^2)^2 - 4x^2}}}{2}\right)}$$

(Remark - this can be simplified)

Homographic transformations

These are conformal transformations of the form

$$z' = \frac{az+b}{cz+d}$$

$$\frac{d}{dz} = \frac{a}{cz+d} - \frac{(ad-b)(cz+d)}{(cz+d)^2} = \frac{ad-cb}{(cz+d)^2}$$

we require $ad-cb \neq 0$ which ensures that $\frac{d}{dz} \neq 0$
These can be inverted

\[(cz + d)z' = az + b\]

\[z(cz' - a) = b - dz'\]

\[z = \frac{-dz' + b}{cz' - a}\]

This is another homographic transformation with

\[a' = a, \quad b' = b, \quad c' = c, \quad d' = -a\]

\[ad' - bc' = (a)(-c) - bc = ad - bc \neq 0\]

This shows that the inverse of a homographic transformation is a homographic transformation.

Next consider

\[z' = \frac{az + b}{cz + d}\]

\[z'' = \frac{a'z' + b'}{c'z' + d'}\]

Combining these

\[z''(z) = \frac{a'(\frac{az + b}{cz + d}) + b'}{c'\left(\frac{az + b}{cz + d}\right) + d'} = \]
\[
\begin{align*}
\frac{a' a z + a' b}{c' a z + c' b + d' c z + d' d} & = \\
\frac{(a a' + b' c) z + (c' a + b' d)}{(c' a + d' c) z + (c' b + d' d)}
\end{align*}
\]

\[
\begin{align*}
a'' & = a c' + b' c \\
b'' & = a' b + b' d \\
c'' & = c' a + d' c \\
d'' & = c' b + d' d
\end{align*}
\]

Consider matrix multiplication

\[
\begin{pmatrix}
a' & b' \\
c' & d'
\end{pmatrix}
\begin{pmatrix}
a b \\
c d
\end{pmatrix}
\]

\[
\begin{pmatrix}
a' a + b' c & a' b + b' d \\
c' a + d' c & c' b + d' d
\end{pmatrix}
\]

Note

\[
(aa' + b' c)(c' b + d' d) - (a' b + b' d)(c' a + d' c)
\]

\[
= a' c' a b + a' d' a d + b' c' c b + d' b' c d + \]

\[
- a' c' b a - a' d' b c - b' c' a d - b' d' d c
\]

\[
a' d' (a d - b c) + b' c' (c b - a d)
\]

\[
= (a d - b c) (a' d' - b' c') \neq 0
\]

This shows that (1) the composition of 2 homographic transformations is a homographic transformation.
If \( a = a = 1 \) and \( b = b = 0 \) then 
\[ z' = \frac{z}{1} = z \]
so the identity is a homographic transformation.

**Definition:** A group is a set \( G \) with a product \( \cdot \) satisfying:
- \( g_1, g_2 \in G \) \( \implies \) \( g_1 \cdot g_2 \in G \) \( \text{closed} \)
- \( e \in G \) \( \implies \) \( e \cdot g = g \cdot e = g \) \( \text{identity} \)
- \( g \in G \) \( \implies \) \( g^{-1} \in G \) \( \text{inverse} \)
- \( g_1, g_2, g_3 \in G \) \( \implies \) \( g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3 \) \( \text{associative} \)

The set of homographic transformations forms a group under composition.

The identification 
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \frac{az + b}{cz + d}
\]
is a 1-1 correspondence between homographic transformations and \( 2 \times 2 \) complex matrices with non-zero determinant that preserves group multiplication.
The 2 groups are isomorphic.

\[ \text{GL}(2, \mathbb{C}) \cong \text{homographic transformations} \]

Remarks:

\[ z = r e^{i \phi} \]
\[ \frac{1}{z} = \frac{1}{r} e^{-i \phi} \] (inversions)
\[ cz = cr e^{i \phi} \] (scale transformations)
\[ z \to z + a \] (translations)

Each of these transformations is a homographic transformation.

\[ \frac{1}{z} \quad a=0 \quad b=1 \quad c=1 \quad d=0 \]
\[ cz \quad a=c \quad b=0 \quad c=0 \quad d=1 \]
\[ a=1 \quad b=a \quad c=0 \quad d=1 \]

Each of the above 3 transformations map circles in the complex plane to circles. It is possible to show that these three transformations generate all homographic transformations.
This means homomorphic transforms generally transform circles to circles.

The isomorphism means that any property of $GL(2\mathbb{C})$ translates to a property of homomorphic transforms.

In $GL(2\mathbb{C})$, $ad - bc \neq 0$. If we restrict $abcd$ so $ad - bc = 1$, it is clear since we showed that

$$\det(M_1 M_2) = \det M_1 \det M_2$$

$$\det(M_1^* M_1) = \det I = 1 \Rightarrow \det M^* = 1$$

It follows that the $2 \times 2$ complex matrices with $\det M = 1$ is a subgroup of $GL(2\mathbb{C})$. This is a subset of $GL(2\mathbb{C})$ with all of the group properties.

Consider

$$X = \begin{pmatrix} ct + 2 & x - iy \\ x + iy & ct - 2 \end{pmatrix}$$

$$\det X = c^2 t^2 - x^2 - y^2 - 2^2$$
If \( A \in SL(2, \mathbb{C}) \) - i.e. the subgroup of \( SL(2, \mathbb{C}) \) with \( \det 1 \)

\[
X' = AXA^+ \Rightarrow A_{ij} = A^*_{ij}
\]

\[
\det X' = \det A \det X \det A^+
= 1 \cdot ((ct)^2 - x^2y^2 - z^2) 1
\]

This shows that \( X' = AXA^+ \)

\[
X' = \begin{pmatrix} ct' + z' & x' - iy' \\ x' + iy' & ct' - z' \end{pmatrix}
\]

is a linear transformation that preserves \( c^2t^2 - x^2y^2 \) which defines Lorentz transformations.

The subgroup of homography transformations satisfying \( ad - bc = 1 \) is isomorphic to \( SL(2, \mathbb{C}) \) which is a subgroup of the group of Lorentz transformations.
The Cauchy-Goursat Theorem

**Theorem** Let \( C \) be a piecewise regular closed curve in the complex plane. Assume \( f(z) \) is analytic on \( C \) and at all points enclosed by \( C \).

Then \( \oint_C f(z) \, dz = 0 \)

(regular \( C = C(a) \) differentiable)

*Analyticity on \( C \) and in the interior is essential*

*Cauchy proved this assuming \( \frac{df}{dz} \) was continuous, Goursat proved the theorem without this assumption*

**Simple Cases**

\[ z^n = \frac{1}{n+1} \frac{d}{dz} z^{n+1} \]

\[ \oint_C z^n \, dz = \frac{1}{n+1} \left( C(b) - C(a) \right) = 0 \]

Since the curve is closed

Since integration is linear this extends to polynomials

\[ \oint_C \sum a_n z^n = \sum \oint_C a_n \left( C(b) - C(a) \right) = 0 \]