1. Consider the complex valued function of the complex variable $z$

$$f(z) = e^{z^2}$$

a. Find the real and imaginary parts of this function.

Differentiating gives

$$f(z) = e^{x^2 - y^2 + 2ixy} = e^{x^2 - y^2} (\cos(2xy) + i \sin 2xy)$$

$$Re(f(z)) = e^{x^2 - y^2} \cos(2xy) \quad Im(f(z)) = e^{x^2 - y^2} \sin(2xy)$$

b. Show that the real and imaginary parts of this function satisfy the Cauchy-Riemann equations.

$$\frac{\partial u}{\partial x} = e^{x^2 - y^2} (2x \cos(2xy) - 2y \sin(2xy))$$

$$\frac{\partial u}{\partial y} = e^{x^2 - y^2} (-2y \cos(2xy) - 2x \sin(2xy))$$

$$\frac{\partial v}{\partial x} = e^{x^2 - y^2} (2x \sin(2xy) + 2y \cos(2xy))$$

$$\frac{\partial v}{\partial y} = e^{x^2 - y^2} (-2y \sin(2xy) + 2x \cos(2xy))$$

Comparing lines 1 and 4 and lines 2 and 3 gives

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

C. Show that the real part of this function satisfies Laplace's equation.

From part b

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} \frac{\partial}{\partial x} = -\frac{\partial}{\partial y} \frac{\partial u}{\partial y} = -\frac{\partial^2 u}{\partial y^2}$$

Putting everything on the same side of the equation gives

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
This can be done by brute force as well:

\[
\frac{\partial^2 u}{\partial x^2} = 2x e^{x^2 - y^2}(2x \cos(2xy) - 2y \sin(2xy)) + e^{x^2 - y^2}(2 \cos(2xy)) +
\]

\[
e^{x^2 - y^2}(-4xy \sin(2xy) - 4y^2 \cos(2xy)) =
\]

\[
e^{x^2 - y^2}(4x^2 + 2 - 4y^2) \cos(2xy) + (-8xy) \sin(2xy)
\]

\[
\frac{\partial^2 u}{\partial y^2} = -2ye^{x^2 - y^2}(-2y \cos(2xy) - 2x \sin(2xy)) +
\]

\[
e^{x^2 - y^2}(-2 \cos(2xy)) + e^{x^2 - y^2}(4yx \sin(2xy) - 4x^2 \cos(2xy)) =
\]

\[
e^{x^2 - y^2}(4y^2 - 2 - 4x^2) \cos(2xy) + (8xy) \sin(2xy) = \frac{\partial^2 u}{\partial y^2}
\]

adding these terms gives 0

d. Is \(f(z)\) an entire function?

Yes - since it is the composition of two entire functions.

2. Consider the function

\[ f(z) = \frac{1}{z(z^2 + 4)} \]

a. Find the poles of \(f(z)\).

Factoring the denominator shows that \(f(z)\) has three poles at 0, \(2i\) and \(-2i\)

b. Find the residue of each pole of \(f(z)\).

\[ f(z) = \frac{1}{z(z + 2i)(z - 2i)} \]

The residues are: \(r_0 = \frac{1}{4}, r_{2i} = (\frac{1}{2i})(\frac{1}{4i}) = -\frac{1}{8}, r_{-2i} = (\frac{-1}{2i})(\frac{-1}{4i}) = -\frac{1}{8}\)

c. Find the Laurent series for \(f(z)\) about \(z = 0\) that converges for 0 < \(|z|\) < 2.

\[ f(z) = \frac{1}{4} \sum_{n=0}^{\infty} (-\frac{1}{z/2})^{2n} = \frac{1}{4} \sum_{n=0}^{\infty} (-\frac{z}{2})^{2n} = \frac{1}{4} \sum_{n=0}^{\infty} \frac{z^{2n-1}}{4^n} \]

d. What is the integral of \(f(z)\) over a counterclockwise circle of radius \(|z| = \frac{1}{2}\)?

Only the pole at \(z = 0\) is in the circle

\[ I = 2\pi i (1/4) = i\pi /2 \]
e. What is the integral of \( f(z) \) over a counterclockwise circle of radius \( |z| = 1 \)?

Only the pole at \( z = 0 \) is in the circle

\[
I = 2\pi i (1/4) = i\pi/2
\]

f. What is the integral of \( f(z) \) over a counterclockwise circle of radius \( |z| = 4 \)?

All three poles are in the circle

\[
I = 2\pi i [(1/4 - 1/8 - 1/8) = 0].
\]

3. Let \( f(z) \) be an entire function.

a. Let \( \gamma(t) \) be a path in the complex plane between \( z_0 \) and \( z \); \( \gamma(0) = z_0 \), \( \gamma(1) = z \). Define

\[
g(z,z_0,\gamma) = \int_0^1 f(\gamma(t)) \frac{d\gamma}{dt} dt
\]

Does \( g(z,z_0,\gamma) \) depend on the path \( \gamma(t) \) between \( z \) and \( z_0 \)? Why?

No, by Cauchy’s theorem because \( f(z) \) is analytic.

b. Does \( g(z,z_0,\gamma) \) depend on \( z_0 \)?

Yes, different choices differ by a constant.

c. Is \( g(z,z_0,\gamma) \) an analytic function of \( z \)? Justify your answer.

Yes, since the value of the integral is independent of the curve, it in independent of how the curve approaches \( z \). Using the definition of derivative

\[
\frac{dg(z)}{dz} = \lim_{\Delta z \to 0} \frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(z') dz' = f(z)
\]

Alternatively, the integral of the convergent Taylor series is

\[
\int \sum_{n=0}^{\infty} a_n z^n = z \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^n
\]

which converges since \( |a_n/(n + 1)| \leq |a_n| \)

4. Residue theorem

a. Consider the integral

\[
\int_0^\infty \frac{\sin(x)}{x(x^2 + a^2)} dx
\]
where \(a\) is real. Express this integral in a form where it can be computed using the residue theorem.

Introduce an \(i\epsilon\) in the denominator
\[
\int_0^\infty \frac{\sin(x)}{x(x^2 + a^2)} \, dx
\]
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{-\infty}^{\infty} \frac{\sin(x)}{(x - i\epsilon)(x^2 + a^2)} \, dx
\]
\[
\frac{1}{4i} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{e^{ix}}{(x - i\epsilon)(x - ia)(x + ia)} \, dx - \frac{1}{4i} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{e^{-ix}}{(x - i\epsilon)(x - ia)(x + ia)} \, dx
\]

b. Show the contours that you would use to compute the integral in part a.
Close first integral in upper half plane counter clockwise, second integral in lower half place clockwise - notice the sign change in the second integral

c. Evaluate the integral in part a. using the contours in part b.
\[
(2\pi i) \left[ \frac{1}{4i} \left( \frac{1}{a^2} + \frac{e^{-a}}{(-2a^2)} \right) + \frac{1}{4i} \left( \frac{e^{-a}}{(-2a^2)} \right) \right] = \pi / (2a^2)(1 - e^{-a})
\]
d. Use the residue theorem to evaluate the infinite sum
\[
\sum_{n=0}^{\infty} \frac{1}{n^2 + \pi^2}
\]
\[
0 = \int \frac{\pi \cot(z\pi)}{z^2 + \pi^2} =
\]
\[
2\pi i \left( \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + \pi^2} + \frac{\pi \cot(i\pi^2)}{2i\pi} + \frac{\pi \cot(-i\pi^2)}{2(-i\pi)} \right) =
\]
\[
2\pi i \left( \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + \pi^2} - \frac{\pi \coth(\pi^2)}{2\pi} - \frac{\pi \coth(\pi^2)}{2\pi} \right) = 0
\]
Recall \(\cot(iz) = \cos(iz) / \sin(iz) = \cosh(z) / (i \sinh(z))\); Dividing the full sum by 2 and adding \(\frac{1}{2\pi^2}\) gives the result
\[
\sum_{n=0}^{\infty} \frac{1}{n^2 + \pi^2} = \frac{1}{2\pi^2} + \frac{\coth(\pi^2)}{2}
\]