29:4761 - final exam

1. Let \( z = (a + ib)^{e+i\theta} \)
   a. Calculate the real and imaginary parts of \( z \).
   b. Calculate the modulus and argument of \( z \).
   c. Calculate \( \ln(z) \).

2. Let \( f(x, y) = u(x, y) + iv(x, y) \) be analytic.
   a. Show that \( u(x, y) \) and \( v(x, y) \) are solutions of Laplace's equation in 2 dimensions.
   b. Assume that \( r(x, y) \) and \( s(x, y) \) are both solutions of Laplace's equation. Is \( g(x, y) = r(x, y) + is(x, y) \) analytic.
   c. Assume that \( h(z) \) is analytic and constant for all \( z \) satisfying \( |z| = R \) = constant. What can you say about \( |h(z)| \)? Justify your answer.

3. Let \( a \) and \( b \) be real numbers. Evaluate the integrals

\[
I(a) = \int_0^\infty \frac{\sin(x)}{x(x^2 + a^2)}
\]

\[
I(a, b) = \int_0^\infty \frac{1}{(x^2 + b^2)(x^2 + a^2)}
\]

4. Let \( H \) be a hermitian matrix \( (H = H^\dagger) \).
   a. Show that the eigenvalues of \( H \) are real.
   b. Show that the eigenvectors of \( H \) corresponding to distinct eigenvalues are orthogonal.
   c. Assume that in some basis \( H \) is also real. Show that the components of the eigenvectors in that basis can be chosen to be real.

5. Let \( H \) be a hermitian matrix, \( H = H^\dagger \) and let \( a \) be real.
   a. Show that \( (H + ia)(H - ia) \) is a positive operator.
   b. Show \( e^{-H} \) is positive.
   c. Is \( e^{-H}(H + ia)(H - ia) \) positive?

6. Consider a matrix \( J \) with eigenvalues \( \lambda = 1, 0, -1 \).
   a. What is the characteristic polynomial of \( J \)?
   b. Find projections

\[
P_1 = |1\rangle\langle 1| \quad P_0 = |0\rangle\langle 0| \quad P_{-1} = |-1\rangle\langle -1|
\]

expressed as polynomials in \( J \).
   c. Calculate \( e^{i\theta J} \) as a finite degree polynomial in \( J \) (here \( \theta \) is real).
Final exam solutions

(a) \((a+ib)^{c+id} = e^{\ln(a+ib)^{c+id}} = e^{(c+id)\ln(a+ib)}\)

\[\text{let } a+ib = re^{i\phi} = r\cos\phi + i r\sin\phi.\]

\[(a+ib)^{c+id} = e^{(c+id)\ln re^{i\phi}} = e^{(c+id)(\ln r + i(\phi+2\pi n))}\]

\[e^{\ln r^c - d(\phi+2\pi n)} i(d\ln r + c\phi + 2\pi n c)\]

\[r^c e^{-d(\phi+2\pi n)} \cos(d\ln r + c\phi + 2\pi n c) + i r^c e^{-d(\phi+2\pi n)} \sin(d\ln r + c\phi + 2\pi n c)\]

real part:
\[r^c e^{-d(\phi+2\pi n)} \cos(d\ln r + c\phi + 2\pi n c)\]

imaginary part
\[r^c e^{-d(\phi+2\pi n)} \sin(d\ln r + c\phi + 2\pi n c)\]

(b) \[|Z| = r^c e^{-d(\phi+2\pi n)}\]

\[\Theta = d \ln r + c\phi + 2\pi n c\]

(c) \(\ln Z = (c+id)\ln(a+ib) =\]

\[= (c+id)\ln re^{i(\phi+2\pi n)}\]

\[= (c+id)(\ln r + i(\phi+2\pi n))\]

\[= c\ln r - d(\phi+2\pi n) + i(d\ln r + c(\phi+2\pi n))\]
2 analytic
\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \]
\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 v}{\partial y^2} = 0 \]
\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u = 0 \]
\[ \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 v}{\partial y^2} = 0 \]
\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) v = 0 \]

2b not necessarily - we also need
\[ \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} = 0 \]
which is a consequence of the Cauchy Riemann equation's

2c \( f(z) \) is constant - this is because (i) all derivatives can be computed along the circle - but they are all 0, so the taylor expansion has 1 term - which is a constant
\[\int_{0}^{\infty} \frac{\sin x}{x(a^2+x^2)} \, dx = \int_{-\infty}^{\infty} \frac{\sin x}{(x-i\varepsilon)(x-i\varepsilon)^{\frac{1}{2}}} \, dx = \frac{1}{2} \cdot \frac{1}{2i} \int_{-\infty}^{\infty} \left[ \frac{e^{ix}}{(x-i\varepsilon)(x-i\varepsilon)(x+ia)} - \frac{e^{-ix}}{(x-i\varepsilon)(x-i\varepsilon)(x+ia)} \right] \, dx \]

\[= \frac{1}{4i} 2\pi i \left( \frac{1}{a^2} + \frac{e^{-a}}{ia(2ia)} + \frac{e^{-a}}{(-ia)(-2ia)} \right) \]

\[= \frac{\pi}{2a^2} \left( 1 - \frac{1}{2} e^{-a} - \frac{1}{2} e^{-a} \right) = \frac{\pi}{2a^2} \left( 1 - e^{-a} \right) \]

\[\int_{0}^{\infty} \frac{dx}{(x^2+b^2)(x^2+c^2)} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(x^2+b^2)(x^2+c^2)} = \frac{2\pi i}{2} \left\{ \frac{1}{2ia} \frac{1}{b^2-a^2} + \frac{1}{2ib} \frac{1}{a^2-b^2} \right\} \]

\[= \frac{\pi}{2} \frac{1}{a^2-b^2} \frac{1}{ab} (a-b) = \frac{\pi}{2} \frac{1}{(a+b)ab} \]

\[\langle n_1|n_1 \rangle = \lambda_n \langle n_1 \rangle \]

\[\langle n_1|n_1 \rangle^* = \langle n_1^H|n_1 \rangle = \langle n_1|n_1 \rangle = \lambda_n^* \langle n_1 \rangle \]

\[\lambda_n - \lambda_n^* \langle n_1 \rangle = 0 \quad \text{since} \quad \langle n_1 \rangle \neq 0 \quad \lambda_n = \lambda_n^* \]
\( \langle n | H | m \rangle = \lambda_m \langle n | m \rangle \)
\( \langle n | H | m \rangle^* = \langle m | H | n \rangle = \lambda_n \langle m | n \rangle \)
\( \langle n | H | m \rangle = \langle n | H | m \rangle^* = \lambda_n^* \langle n | m \rangle = \lambda_n \langle n | m \rangle \)

\( \Rightarrow \langle n | m \rangle (\lambda_m - \lambda_n) = 0 \)
if \( \lambda_n \neq \lambda_m \)
\( \langle n | m \rangle = 0 \)

6. \( H_{mn} V_n = \lambda_n V_m \)
\( H_{mn} V_n^* = \lambda_n V_m^* \quad (H_{mn} \text{ real}) \)
\( \omega_n = \frac{1}{2} (V_n + V_n^*) \)
\( \tilde{\omega}_n = \frac{i}{2} (V_n^* - V_n) \)
are both real and both eigenvectors.

5b. \( (H-i\alpha)(H+i\alpha) = (H+i\alpha)^+ (H+i\alpha) \)

\( \langle c | (H-i\alpha)(H+i\alpha) | c \rangle = \| (H+i\alpha) c \|^2 > 0 \)
\( \Rightarrow H \text{ is positive} \)

b. \( e^{-H} = e^{-H/2} e^{-H/2} = e^{-H/2} - H/2 \)
\( \langle c | e^{-H} | c \rangle = \| e^{-H/2} c \|^2 > 0 \)
Yes

\[ e^{-\frac{H}{2}} (H+i\alpha) (H-i\alpha) = (H-i\alpha)^{\dagger} e^{-\frac{H}{2}} e^{H} (H-i\alpha) \]

\[ \langle c| e^{-(H+i\alpha)(H-i\alpha)c} = \langle 1 \rangle e^{-H/2} (H-i\alpha)c \rangle \|^2 \geq 0 \]

\[ (\lambda-1)\lambda (\lambda+1) = \rho (\lambda) \]

\[ \langle 1 \rangle \langle 1| = \frac{J-0}{1-0} \frac{J-(-1)}{1-(-1)} = \frac{J^2+J}{2} \]

\[ \langle 0 \rangle \langle 0| = \frac{J-1}{0-1} \frac{J+1}{0+1} = \frac{J^2-1}{-1} = 1-J^2 \]

\[ \langle -1 \rangle \langle -1| = \frac{J-1}{-1-1} \frac{J-0}{-1-0} = \frac{J^2-J}{2} \]

\[ e^{i\theta} = e^{i\theta} \frac{J^2+J}{2} + (1-J^2) + \frac{J^2-J}{2} e^{-i\theta} \]

\[ = J^2 (\cos \theta - 1) + i J \sin \theta + 1 \]