1. Let $f(z) = e^{2z}$. Let $C$ be the curve in the complex plane that starts at the origin, $z = 0$, goes along the positive real axis to the point $z = 2$, and then proceeds in a straight line in positive imaginary direction from $z = 2$ to $z = 2 + 3i$. Calculate the contour integral

$$\int_C f(z)dz$$

Check your answer by noting that

$$f(z) = \frac{1}{2} \frac{df}{dz}(z)$$

2. Let $f(z)$ be analytic and assume that $f(z)$ is constant on the line segment $0 < x < 1$. Show that $f(z)$ must be constant.

3. Show that any real valued analytic function is constant.

4. Use Darboux's theorem to put a bound on the integral

$$\int_C \sin(z)dz$$

where $C$ is the circle $|z| = 5$.

5. If $C$ is the circle $|z - z_0| = r > 0$, calculate the contour integral (in the counter-clockwise direction) of

$$\int_C \frac{dz}{z}$$

Show that the result is independent of $r$.

6. If $f(z)$ is analytic and non-vanishing in a region $R$, and continuous in $R$ and its boundary, show that $|f|$ assumes its minimum and maximum values on the boundary of $R$. 
Solutions: Set #3

1. \[ f(z) = e^{2z} = e^{2x} e^{2iy} = e^{2x} (\cos 2y + i \sin 2y) \]

\[ \int f(z) = \int_0^2 dx \ e^x (\cos (2x) + i \sin (2x)) + \int_0^3 idy \ e^{2iy} \]

\[ = \frac{1}{2} (e^4 - 1) + i e^4 \frac{1}{2i} (e^{6i} - 1) \]

\[ = \frac{1}{2} (e^4 - 1) + \frac{1}{2} e^4 (\cos 6 + i \sin 6 - 1) \]

Check:

\[ \frac{df}{dz} = \frac{1}{2} e^{2z} \left[ \frac{d}{dz} \right]_{z=0} (e^{2z} + 3i e^{3iz}) \]

\[ = \frac{1}{2} (e^4 - 1) + \frac{1}{2} e^4 (\cos 6 + i \sin 6 - 1) \]

2. Since the complex derivatives can be computed along any path choose \( z_0 \in (0, 1) \) and differentiate with respect to \( x \).

It follows that \( \frac{d^n f(z)}{dz^n} (z_0) = 0 \). Since \( f(z) \) can be represented by a locally convergent Taylor series it must be constant.
\(3\) Assume \(f(z)\) is real, with
\[
f(z) = u(z) + i \, v(z)
\]
reality means \(v(z) = 0\). The Cauchy-Riemann equations give
\[
\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0
\]
\[
\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0
\]
the only compatible analytic solution is \(f(z) = \text{constant}\).

\(4\) By Danboux's theorem
\[
\left| \int_c \sin z \, dz \right| \leq (2\pi R) \max_{z = \text{Re}^i\theta} |\sin z|
\]
The non-trivial part is to note that \(\sin z\) is not periodic.

If we let \(z = x + iy = R\cos \theta + i R\sin \theta\)
\[
|\sin z| = \left( (\sin z)(\sin z)^* \right)^{\frac{1}{2}} = \left( \int \frac{1}{2i} (e^{i(x+iy)} - i^{(x+iy)}) \, \frac{-1}{2i} (e^{-i(x-iy)} - i^{(x-iy)}) \right)^{\frac{1}{2}} = \left( \frac{1}{4} (e^{-2y} + 2y - e^{-2y}) \right)^{\frac{1}{2}} = \left( \frac{1}{2} \cos 2y - \frac{1}{2} \cos (2x) \right)^{\frac{1}{2}}
\]
For \( x = R \cos \theta \quad y = R \sin \theta \)

\[
|\sin z|^2 = \frac{1}{2} (\cosh (R \sin \theta) - \cos (R \cos \theta))
\]

we need to find a \( \theta \) that maximizes this quantity. The critical values of solutions of

\[
\frac{d}{d\theta} (|\sin z|^2) = 0 = \frac{1}{2} (\sin (R \sin \theta) \cdot R \cos \theta - \sin (R \cos \theta) \cdot R \sin \theta)
\]

this vanishes when \( \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \)

It is clear that there must be a minimum between 2 maxima.

By inspection \( \theta = 0 \) value is less than \( \theta = \frac{\pi}{2} \) so \( \frac{\pi}{2} \) and \( \frac{3\pi}{2} \) must be the maxima; both give

\[
|\sin z|_{\max}^2 = \frac{1}{2} (\cosh R - 1)
\]

assuming \( \cosh R - 1 \geq 1 - \cos R \) which is true \( \ln R = 5 \).

\[
\therefore |\int \sin z \, dz| \leq (2\pi R) \sqrt{\frac{1}{2} (\cosh R - 1)}
\]
\[ \oint \frac{dz}{z} = \int_{0}^{2\pi} iRe^{i\phi} \, d\phi = 2\pi i \]

and the result is independent of \( R \).

Consider

\[ f(z) = \frac{1}{2\pi i} \oint \frac{f(z')}{z'-z} \, dz' \]

asssume \( f(z) \) has a local maximum

use a circular path

\[ |f(z)| \leq \frac{2\pi R}{2\pi R} \max_{\phi \in \text{circle}} |f(Re^{i\phi})| \]

\[ \leq \max_{\phi} |f(Re^{i\phi})| \]

\[ = |f(z')| \text{ in some } z' \text{ in the boundary} \]

This is true for any \( R \) so this is an infinite sequence of points \( z', z'', \ldots \rightarrow z \) and \( |f(z')| \leq |f(z)| \)

this means that \( z \) cannot be a local maximum.
For the minimum let

\[ g(z) = \frac{1}{f(z)} \]

\[ |g(z)| \leq |g(z')| \text{ in some } z' \]
on a circle of radius R

\[ |f(z)| \geq |f(z')| \]

Again - since the result is independent of R, there is a sequence \( z_n \to 0 \) when
\[ |f(z_n)| \leq |f(z)| \text{, so } |f(z)| \text{ can't be a local minimum} \]