1. Consider

\[ f(z) = \int_0^1 \frac{dx}{x - z} \]

Show that \( f(z) \) is analytic for \( z \not\in [0,1] \)

2. Integrate \( f(z) \) in problem 1 around a square path with vertices at \( x = \pm 2, y = p \pm 2 \).

3. Show that if \( f(z) \) is entire and \( |f(z)| \leq C|z^n| \) for sufficiently large values of \( |z| \), where \( C \) is a constant, then \( f(z) \) must be a polynomial of degree \( \leq n \).

4. Let \( f(t) \) be continuous for \( t \in [a, b] \) with \( a \neq b \) finite. Show

\[ F(z) = \int_a^b e^{itz} f(t) dt \]

is an entire function. Calculate the derivative.

5. Show that any non-constant polynomial in \( z \) has at least one complex root.

6. Prove that for charge free two-dimensional space the value of the electrostatic potential at any point is equal to the average of the potential over the circumference of any circle centered on the point. You may assume that the potential is the real part of an analytic function (the electrostatic potential in a charge free region is a solution of Laplace's equation).
\[
\frac{df}{dz}(z) = \lim_{\Delta z \to 0} \left[ \int_{0}^{1} \frac{dx}{x-z-\Delta z} - \int_{0}^{1} \frac{dx}{x-z} \right] \frac{1}{\Delta z}
\]

\[
= \lim_{\Delta z \to 0} \int_{0}^{1} dx \frac{x-z-x+z+\Delta z}{(x-z-\Delta z)(x-z)} \frac{1}{\Delta z}
\]

Since \( x-z \neq 0 \) for all \( x \in [0,1] \)

\[
= \lim_{\Delta z \to 0} \int_{0}^{1} \frac{dx}{(x-z)(x-z-\Delta z)}
\]

This exists and is independent of how \( \Delta z \to 0 \) as long as \( z \) is not on the interval \([0,1] \)

\[
= \int_{0}^{1} \frac{dx}{(x-z)^2}
\]

Since the derivative exists for \( z \in [0,1] \), \( f(z) \) is analytic in those \( z \) s.
In this problem we can use Cauchy's theorem to change the path of integration from the square to any curve in the plane that does not intersect \([0,1]\) (see diagram).

\[
\int_{\Box} dz \int_{0}^{1} \frac{dx}{x-z}
\]

For \(z\) on the square \(|\frac{1}{x-z}| \leq 1\) so we can use Cauchy's theorem to change the order of integration:

\[
= \int_{0}^{1} dx \int_{\Box} dz \frac{1}{x-z} = -\int_{0}^{1} dx \int_{\Box} \frac{dz}{z-x}
\]

The integral over the square surrounding \(x\) can be replaced with a circle of radius \(R > 1\) centered at \(Z = x\) \(Z = x + Re^{i\theta}\).
The integral becomes

\[- \int_0^1 \, dx \, \int_0^{2\pi} \frac{i \text{Re} e^{i\phi}}{\text{Re} e^{i\phi}} = - \int_0^1 \, dx \, 2\pi i = -2\pi i\]
Analyticity implies

\[ \frac{d^k f}{dz^k}(z) = \frac{k!}{2\pi i} \int \frac{f(z')}{(z'-z)^{k+1}} \, dz' \]

Let \( z = 0 \) and \( z' = Re^{i\varphi} \), \( dz' = iRe^{i\varphi} d\varphi \)

then

\[ \frac{d^k f}{dz^k} = \frac{k!}{2\pi i} \int \frac{f(Re^{i\varphi})}{R^k e^{-ik\varphi}} iRe^{i\varphi} d\varphi \]

\[ = \frac{k!}{2\pi} \frac{1}{R^k} \int_0^{2\pi} f(Re^{i\varphi}) e^{-ik\varphi} d\varphi \]

From Darboux theorem

\[ \left| \frac{d^k f}{dz^k}(0) \right| < \frac{k!}{2\pi} \frac{1}{R^k} 2\pi CR \]

This is true for any \( k \) and \( R \), for \( k > n \)

\[ \left| \frac{d^k f}{dz^k}(1) \right| < \frac{C k!}{R^{k-n}} \]

which vanishes as \( R \to \infty \)

The means \( \frac{d^k f}{dz^k}(1) = 0 \) for \( k > n \)

By Taylor's Theorem

\[ f(z) = \sum_{k=0}^{n} \frac{1}{k!} \frac{d^k f}{dz^k}(0) z^k + \text{polynomial of degree } \leq n \]
$F(z) = \int_a^b e^{itz} f(t) \, dt$

Consider the series:

$$\sum_{n=0}^{\infty} z^n \int_{\gamma}^{b} \frac{(it)^n}{n!} f(t) \, dt$$

$$a_n = \frac{\max |f(t)|}{a \leq t \leq b}$$

$$T = \max |t|$$

$$|a_n| \leq \frac{1}{n!} (b-a) B T^n$$

It follows that:

$$\left| \sum_{n=0}^{\infty} z^n \int_{\gamma}^{b} \frac{(it)^n}{n!} f(t) \, dt \right| \leq \sum_{n=0}^{\infty} \frac{|z|^n T^n B (b-a)}{n!} = B(b-a) e^{T|z|} < \infty$$

This means that the sum is absolutely convergent for any $z$ in the complex plane by the converse of Taylor's theorem.

It follows that $\frac{dF}{dz}$ has an absolutely convergent power series given by:

$$\frac{dF}{dz} = \int_{\gamma}^{b} e^{itz} f(t) \, dt$$

This is one of many ways to do this.
If \( P(z) \) has no zeroes then
\[
g(z) = \frac{1}{P(z)}
\]
is an entire function; since \( P(z) \) is not constant \( g(z) \) is not constant.

If \( g(z) \) is bounded then
\[
\frac{d g}{d z} = \frac{1}{2\pi i} \oint \frac{g(z')}{(z'-z)^2} \, dz'
\]
\[
\left| \frac{d g}{d z} \right| \leq \frac{1}{2\pi} \frac{2\pi R}{R^2} \max_{|z|=R} g(z)
\]
\[
\leq \frac{b}{R}
\]
where \( R \) is arbitrary. This means \( \frac{d g}{d z} = 0 \) in the complex plane \( \Rightarrow g(z) \) is constant.

This contradicts \( g(z) \) being non-constant. This contradicts the assumption that \( g(z) \) is bounded.

\[
g(z_n) \to 0 \Rightarrow P(z_n) \to 0.
\]
So \( P(z) \) must have a \( 0 \).

\((z_n \to z) \Rightarrow P(z) = 0\)
This is a direct application of the Poisson integral formula

\[ \nabla^2 \phi = 0 \text{ in charge free region} \]

\[ \phi(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} \phi(r', \phi') \frac{R^2 - r^2}{r^2 + r'^2 - 2rr' \cos(\phi - \phi')} \]

which expresses the potential in terms of its value on the circle.