Lecture 4

Integrals of complex functions

The most common cases are integrals over curves in the complex plane.

The curves of interest are complex valued functions of a real argument

\[ y(\lambda) = y_x(\lambda) + iy_y(\lambda) \]

where \( \lambda \) is a real parameter that varies between \( a \) and \( b \).

The integral of a complex function \( f(z) \) along the path \( \gamma \)

\[ \int_\gamma f(z) \, dz \]

like an ordinary integral can be expressed in terms of Riemann sums

\[ \int_\gamma f(z) \, dz = \lim_{\Delta z_i \to 0} \sum f(z_i) \, \Delta z_i \]

The \( \Delta z_i = y(\lambda_i + \Delta \lambda) - y(\lambda_i) = \frac{dy}{d\lambda}(\lambda_i) \Delta \lambda + o(\Delta \lambda)^2 \)

\[ f(z_i) \approx f(y(\lambda_i)) \]
The expression can be replaced by

$$\lim_{\Delta \lambda \to 0} \sum_{i=1}^{N} f(\lambda_i) \left( \frac{d^y(\lambda_i) \Delta \lambda + o(\Delta \lambda^3)}{d\lambda} \right)$$

since $\Delta \lambda \sim \frac{1}{N}$ where $L = b-a$ and $N$ is the number of terms in the sum.

The limit of what remains can be expressed in terms of ordinary integrals

$$\int_{a}^{b} f(\lambda) \frac{d^y}{d\lambda} d\lambda =$$

$$\int_{a}^{b} (U(\lambda) + iV(\lambda)) \left( \frac{d^y}{d\lambda} + i \frac{d^x}{d\lambda} \right) d\lambda$$

$$\int_{a}^{b} (U \frac{d^y}{d\lambda} - V \frac{d^x}{d\lambda}) d\lambda + i \int_{a}^{b} (U \frac{d^x}{d\lambda} + V \frac{d^y}{d\lambda}) d\lambda$$

Examples of paths

$$\gamma(\lambda) = R e^{2\pi i \lambda}, \quad \lambda \in [0, 1]$$

$$= R \cos(2\pi \lambda) + i R \sin(2\pi \lambda)$$
This path is a circle of radius $R$ centered at $z=0$ in the complex plane.

\[ y(z) = \begin{cases} 
10\lambda & 0 \leq \lambda \leq \frac{1}{2} \\
5 + 4i(\lambda - \frac{1}{2}) & \frac{1}{2} \leq \lambda \leq \frac{3}{2}
\end{cases} \]

For the path of example 2 and $f(z) = z^2$

\[
\int_{y} f(z) \, dz = \int_{0}^{\frac{1}{2}} (10\lambda)^2 \cdot 10 \, d\lambda \\
\int_{\frac{1}{2}}^{\frac{3}{2}} (5 + 4i(\lambda - \frac{1}{2})) \cdot 4i \, d\lambda \\
= (10)^3 \times \left(\frac{1}{2}\right)^3 + (20i + 8) \cdot 1 - 16 \frac{1}{2} \left(\frac{9}{4} - \frac{1}{4}\right) \\
(125 - 8) + 20i = 117 + 20i
\]

The result is a complex number.
2. Let \( z(t) = R(\cos(2\pi t) + i \sin(2\pi t)) \)

\( f(z) = z \)

\[
\oint_{C} f(z) \, dz = \int_{0}^{1} \left( R(\cos(2\pi t) + i \sin(2\pi t)) \times 2\pi R (-\sin(2\pi t) + i \cos(2\pi t)) \right) \, dt
\]

\[
\frac{-2\pi R^{2}}{4} \int_{0}^{1} (-2 \sin 2\pi t \cos(2\pi t)) \, dt + \frac{i 2\pi R^{2}}{4} \int_{0}^{1} (\cos(2\pi t) \cos(2\pi t) - \sin(2\pi t) \sin(2\pi t)) \, dt
\]

\[-2\pi R^{2} \int_{0}^{1} \sin 4\pi t + 2i \pi R^{2} \int_{0}^{1} \cos 4\pi t \, dt = 0 + 0
\]

In this example the integral around the closed curve vanishes.

This is because \( z = \frac{d}{dz} \frac{1}{2} z^{2} \) which involves integrating a derivative around a closed path.
Let $Y$ be the same circle as in the previous example, but consider

\[ \int_Y \ln z \, dz = \int_0^1 \ln (Y(z)) \frac{dy}{dx} \, dx. \]

\[
Y(x) = \Re e^{i 2\pi x}
\]

\[
\ln Y(x) = \ln R + i(2\pi (\lambda + n))
\]

\[
\int_0^1 (\ln R + (2\pi (\lambda + n))i) (\Re e^{2\pi i \lambda} - 2\pi i \lambda) \, dx
\]

\[\frac{2\pi i}{R} \ln R \int_0^1 e^{2\pi i \lambda} \, dx +
\]

\[-4\pi^2 n R \int_0^1 e^{2\pi i \lambda} = 4\pi^2 R \int_0^1 \lambda e^{2\pi i \lambda} \, dx
\]

The first two integrals are multiplied by

\[\frac{1}{2\pi i} (e^{2\pi i} - 1) = 0
\]

The last integral is

\[-4\pi^2 R \frac{1}{2\pi i} \frac{d}{dx} \int_0^1 e^{2\pi i \lambda} \, dx
\]

\[2\pi i R \frac{d}{dx} \left( \frac{1}{2\pi i} (e^{2\pi i \lambda} - 1) \right) |_{\lambda = 0}
\]
\[ 2\pi i R \frac{d}{dz} \left( 1 + \frac{2\pi i d}{2!} + \frac{(2\pi i d)^2}{3!} \right) \left( \frac{1}{z} \right) \]

\[-2\pi^2 R\]

which is not 0 - this is because \( \ln z \) is multiple valued. Going around the circle increased the argument by \( 2\pi i R \).

Note that when

\[ f(z) = \frac{dF}{dz} \]

\[ \frac{dF}{da} = \frac{dF}{dz} \frac{dz}{da} = f(z) \frac{dv}{da} \]

integrating

\[ F(y(1)) - F(y(0)) = \int_0^1 f(z) \frac{dv}{da} da \]

this is the usual fundamental theorem of calculus, but even if \( y(1) = y(0) \) this may not vanish if the function is multiple valued.
An important result that we will use a lot is Danboux's theorem, which estimates the maximum size of the integral

\[
\left| \int f(z) \, dz \right| \leq \left| \int_y f(y(\lambda)) \frac{dy}{d\lambda} \, d\lambda \right| \leq \max_{\lambda \in [a,b]} |f(y(\lambda))| \int_a^b \left| \frac{dy}{d\lambda} \right| \, d\lambda
\]

\[
\left| \frac{dy}{d\lambda} \right| = \sqrt{\left( \frac{dy_1}{d\lambda} \right)^2 + \left( \frac{dy_2}{d\lambda} \right)^2} \, d\lambda = \frac{ds}{d\lambda} \, d\lambda = ds
\]

\[
\int ds = \text{length of the curve}
\]

This means that

\[
\left| \int f(z) \, dz \right| < \max_{z \in Y} |f(z)| \cdot L = F \cdot L
\]

for example

\[
\gamma = Re^{i\phi}, \quad \phi: 0 \to 2\pi; \quad f(z) = z^2
\]

\[
\left| \int_s (Re^{2i\phi}) \cdot Re^{i\phi} \, i \, d\phi \right| < R^2 \cdot 2\pi R = 2\pi R^3
\]
Next we come to the very important topic of analytic functions.

**Definition:** A complex valued function $f(z)$ is **analytic** at $z = z_0$ if there is a neighborhood $N_{r_0} z_0$ where

(A) $f(z)$ is **single valued** in $N_{r_0} z_0$.

(B) $f(z)$ has a complex derivative at $z_0$.

**Definition:** The set of points where $f(z)$ is analytic is called the **domain of analyticity**.

**Definition:** A function whose domain of analyticity is the whole complex plane is called an **entire function**.
examples of entire functions are $e^z$, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$.

A point $z_0$ where $f(z)$ is not analytic is called a **singularity**.

A singularity point is **isolated** if there is a neighbourhood of $z_0$ where $f(z)$ is analytic except at $z_0$.

**Example**

$$f(z) = \frac{1}{z(z-2\pi)}$$

has isolated singular points at $0$ and $2\pi$.

**Cylindrical mapping**

For functions $y = f(x)$ of a real value, it is often of interest to solve for $x = g(y)$.
In example in Hamiltonian mechanics

\[ p = \frac{\partial}{\partial \dot{q}} (q, \dot{q}) \]

defines the momentum in terms of the velocity \( \dot{q} \). To express \( H = p \dot{q} - L(q, \dot{q}) \), we need to solve for \( \dot{q} \) in terms of \( p \) and \( q \).

The condition that it is possible to solve for \( x = q(y) \) in a neighborhood of \( y_0 = q(x_0) \) is that

\[ \frac{df}{dx}(x_0) \neq 0 \]

and is continuous in a neighborhood of \( x_0 \).

The same question with the same result holds in complex functions

given \( z_1 = f(z_2) \) and \( z_2 = q(z_1) \)
The result is called the inverse function theorem.

We assume \( z_1 = f(z_2) \)

where \( \frac{df}{dz_2}(z_{2_0}) \neq 0 \) and \( f(z_2) \) is continuous in a neighborhood of \( z_2 \).

Define a new function

\[
R(z_2) = f(z_2) - f(z_{2_0}) - \frac{df}{dz_2}(z_{2_0})(z_2 - z_{2_0})
\]

\[
= z_1 - z_{1_0} - \frac{df}{dz_2}(z_{2_0})(z_2 - z_{2_0})
\]

The existence of the derivative at \( z_{2_0} \) means that

\[
\frac{R(z_1)}{z_2 - z_{2_0}} \to 0
\]

as \( z_2 \to z_{2_0} \), so by choosing \( z_2 \) close to \( z_{2_0} \), we can make

\[
\frac{R(z_2)}{z_2 - z_{2_0}}
\]

as small as desired.
If we discard $R$

\[ z_2 \approx z_{20} + \left( \frac{df}{dz} (z_{20}) \right) (z_1 - z_{10}) \]

the exact expression is

\[ z_2 = z_{20} + \left( \frac{df}{dz} (z_{20}) \right)^{-1} (z_1 - z_{10} - R(z_1)) \]

This is not a solution because $z_0$ appears on both sides of this equation, but $R(z_0)$ can be made small by choosing $z_2$ near $z_0$.

This suggests solving this by successive approximation:

\[ z_2^{(1)} = z_{20} + \left( \frac{df}{dz} (z_{20}) \right) (z_1 - z_{10}) \]

\[ z_2^{(m)} = z_{20} + \left( \frac{df}{dz} (z_{20}) \right)^{-1} (z_1 - z_{10} - R(z_2^{(m-1)})) \]

by choosing $z_1 - z_{10}$ sufficiently small it can be shown that this iteration converges to a function of $z_1$ where

\[ z_2^{(m)} = z_{20} + \left( \frac{df}{dz} (z_{20}) \right)^{-1} (z_1 - z_{10}) \left( 1 - \frac{R(z_2^{(m-1)})}{z_1 - z_{10}} \right) \]
If \( f(z) \) is analytic, each approximation is analytic at \( z_{10} \). Later we will show that the uniform convergence of this iteration gives an inverse function that is also analytic.

When \( \frac{df}{dz}(z) \neq 0 \) then is a neighborhood of \( z_0 \):

\[
z' = f(z) = u(z) + iv(z)
\]

\[
x' = u(xy) \quad y' = v(xy)
\]

defines an invertible mapping

from a neighborhood \( U \) of \( z_0 \) to one of \( V \) of \( z_0' = f(z_0) \).

In general conformal mappings are transformations of \( \mathbb{N} \) dimensional spaces into themselves that preserve angles.

The uniqueness of the complex derivative can be used to show that complex mappings are conformal.
Let \( f(z) = f(x + iy) = u(x + iy) + iv(x + iy) \)

assume \( \frac{df}{dz}(z_0) \neq 0 \) and let

\( \gamma_1(t) = x_1(t) + iy_1(t) \)

\( \gamma_2(t) = x_2(t) + iy_2(t) \)

intersect at \( z_0 \) at \( t = 0 \)

we write

\( \gamma_1(t) = z_0 + r_1(t) e^{i\phi_1(t)} \)

\( \gamma_2(t) = z_0 + r_2(t) e^{i\phi_2(t)} \)

the image of these curves is

\( \gamma_1'(t) = f(\gamma_1(t)) = z_0' + r_1'(t) e^{i\phi_1'(t)} \)

\( \gamma_2'(t) = f(\gamma_2(t)) = z_0' + r_2'(t) e^{i\phi_2'(t)} \)

\( \frac{d\gamma_2'}{dz}(z_0) = \lim_{t \to 0} \frac{\gamma_2'(t) - f(z)}{\gamma_2(t) - z_0} = \frac{r_2'(t)}{r_1(t)} e^{i(\phi_2'(t) - \phi_1(t))} \)
In this case we computed the derivative along \( \gamma_1 \) we get the same result by computing along \( \gamma_2 \)

\[
\frac{dz'}{dz}(z) = \lim_{\epsilon \to 0} \frac{\gamma_1'(\epsilon) - H(z)}{\gamma_2'(\epsilon) - z} = \frac{r_2'(v)}{r_1'(v)} e^{i(\phi_2'(v) - \phi_1'(v))}
\]

Equating these derivatives gives:

\[
\frac{r_1'(v)}{r_1(v)} = \frac{r_2'(v)}{r_2(v)} = \frac{\phi_1'(v) - \phi_2'(v)}{\phi_1(v) - \phi_2(v) + 2n\pi}
\]

\[
\phi_1'(v) - \phi_2'(v) = \phi_1(v) - \phi_2(v) + 2n\pi
\]

In this case the \( 2n\pi \) leaves the angle between the amplitude unchanged.

\[z' = f(z)\] defines a conformal transformation in a neighborhood of \( z \) where \( \frac{df}{dz}(z) \neq 0 \) and \( f(z) \) is analytic.
\[ x' = u(x, y) \quad \quad x = u'(x', y') \]
\[ y' = v(x, y) \quad \quad y = v'(x', y') \]

The Cauchy-Riemann equations give:
\[ dx' = \frac{\partial u}{\partial x} \, dx + \frac{\partial u}{\partial y} \, dy \]
\[ dy' = -\frac{\partial v}{\partial y} \, dx + \frac{\partial v}{\partial x} \, dy \]

Solving for \( dx \) and \( dy \):
\[ dx = \frac{dx'}{\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \left( \frac{\partial v}{\partial y} \right)^2} \]
\[ dy = \frac{dy'}{\left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \left( \frac{\partial u}{\partial x} \right)^2} \]

Note: \( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 = \left( \frac{\partial x'}{\partial x} \right)^2 \neq 0 \) we can also see that:

\[ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'} \left( \frac{\partial x'}{\partial x} \right)^2 = \frac{\partial u}{\partial x'} \]
\[ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y} \left( \frac{\partial y'}{\partial y} \right)^2 = -\frac{\partial v}{\partial y'} \]
This shows that the inverse functions also satisfy the Cauchy Riemann equations.

If \( f(z) = U(z) + iV(z) \) has \( u \) and \( v \) as solutions of the 2 dimensional Laplace equation, then

\[
g(z') = f(z(z')) = U'(z') + iV'(z')
\]

gives another solution of Laplace equation, while the boundary conditions on \( z \) are replaced by boundary conditions on \( z' \).

If we find \( U(x'y') \) with simple boundary conditions, the get mapped to another solution \( U(x'y) \) is more complex boundary conditions.