Application of conformal mapping

Let $V$ denote a 2 dimensional $\frac{1}{4}$ plane $x,y$ both $> 0$

equations:

consider a small area $\Delta A$ in $V$. The energy in that area is

$$\Delta E = cT\Delta A$$

where $c$ is the heat capacity. We assume that $V$ is insulated so energy can be lost or gained only at the boundary. Here $T$ is the absolute temperature.

The total energy in $V$

$$E = \int dE = \int cT\,dA$$

$$\frac{dE}{dt} = \int_V c \frac{\partial T}{\partial t}\,dA = -\oint_{\partial V} \vec{F} \cdot d\vec{n}\,dr$$

where $\vec{F}$ is the heat current

$$\vec{F} = -k \nabla T$$

where $k$ is the conductivity.

Combining these results - the integral $dA$ is over the boundary of $V$
\[ S_v \frac{\partial T}{\partial t} \, dA = - \int \nabla \cdot d\mathbf{s} \cdot n \, ds \]
\[ = -k \int \nabla T \cdot d\mathbf{s} \cdot n \, ds \]

Using the divergence theorem from vector calculus:
\[ = k \int \nabla \cdot d\mathbf{s} = 0 \]

Putting these together:
\[ \int \left( c \frac{\partial T}{\partial t} - k \nabla^2 T \right) \, dA = 0 \]

This is called the continuity equation, it represents conservation of energy, since it is true for any volume:
\[ c \frac{\partial T}{\partial t} - k \nabla^2 T = 0 \]

When the system reaches thermal equilibrium, \( \frac{\partial T}{\partial t} = 0 \), and the temperature is a solution of Laplace's equation:
\[ \nabla^2 T = 0 \]
we assume the temperature is kept at 0 at x=0 all y

\[ T(xy) = T(0y) = 0 \]

we assume

\[ T(x0) = 1 \quad x > 1 \]

\[ \frac{\partial T}{\partial y}(x0) = 0 \quad 0 \leq x \leq 1 \]

\[ \text{insulating} \quad - \text{no current in the y direction} \]

Let \[ z = x + iy \quad z' = x' + iy' \]

consider the mapping defined by the analytic function

\[ z = \sin z' = \sin(x' + iy') \]

\[ = \sin x' \cosh y' + i \cos x' \sinh y' \]

\[ x = \sin x' \cosh y' \]

\[ y = \cos x' \sinh y' \]

we consider the equivalent boundary conditions in the primed coordinates

\[ x=0 \Rightarrow x'=0 \]

\[ y=0 \quad 0 \leq x \leq 1 \Rightarrow y'=0, \quad x' \in [0, \frac{\pi}{2}] \]

\[ y=0 \quad x > 1 \quad x'=\frac{\pi}{2} \quad y' > 0 \]
we look for a solution of Laplace's equation that satisfies the transformed boundary conditions.

Consider \( f(z') = \frac{2}{\pi} z' = \frac{2}{\pi} (x' + iy') \)

at \( y' = 0 \)

\[
\begin{align*}
  u(x'y') &= \frac{2}{\pi} x' \\
  u(0, y') &= 0 \\
  u(0, \frac{\pi}{2}) &= \frac{\pi}{2}, \frac{2}{\pi} = 1 \\
  u(u0) &= x' \frac{\pi}{2} \quad 0 \rightarrow 1 \quad (x' < \frac{\pi}{2}) \quad \frac{\partial u}{\partial x'} = 0 \text{ is BC}
\end{align*}
\]

to solve the original problem

we need to express

\[
  u(x'y') = u(x'(xy), y(xy)) = \ldots
\]

\( \text{to invent these} \)

\[
\begin{align*}
  \frac{x^2}{\sin^2 x'} &= \cosh^2 y' \\
  \frac{y^2}{\cos^2 x'} &= \sinh^2 y'
\end{align*}
\]
subtracted these give:

\[
\frac{x^2}{\sin^2 x'} - \frac{y^2}{\cos^2 x'} = \cosh^2 y' - \sinh^2 y' = 1
\]

\[
\frac{x^2}{\cosh^2 y'} + \frac{y^2}{\sinh^2 y'} = \sinh^2 x' + \cos^2 x' = 1
\]

to solve for \( x' \):

\[
\frac{x^2}{\sin^2 x'} - \frac{y^2}{1 - \sin^2 x'} = 1
\]

\[
(1 - \sin^2 x') x^2 - \sin^2 x' y^2 = \sin^2 x' - \sin^2 x'
\]

\[
\sin^2 x' - \sin^2 x' (-x^2 - y^2 - 1) + x^2 = 0
\]

This is a quadratic equation in \( \sin^2 x' \):

\[
\sin^2 x' = \frac{1 - x^2 - y^2}{2} \pm \frac{1}{2} \sqrt{(1 - x^2 - y^2) - 4x^2}
\]

This has 2 roots - we introduced one by squaring - for \( x = y = 0 \)

\( x' = y' = 0 \) which is consistent with the lower sign

\[
x' = \left( \sin^{-1} \left( \sqrt{\frac{1 - x^2 - y^2}{2} - \frac{1}{2} \sqrt{(1 - x^2 - y^2)^2 + 4x^2}} \right) \right)
\]

The solution to the BV problem is

\[
u(x, y) = \frac{2}{\pi} x'
\]

\[
= \frac{2}{\pi} \left( \sin^{-1} \left( \sqrt{\frac{1 - x^2 - y^2}{2} - \frac{1}{2} \sqrt{(1 - x^2 - y^2)^2 + 4x^2}} \right) \right)
\]