Lecture 1

Text: Mathematics for Physicists
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Lecture notes, homework, etc
http://homepage.physics.uiowa.edu/~wpolyzou/phys4761

Can also access via class web pages

2 hour exams, 1 final, homework due every week - on Fridays

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1. General math concepts
2. Complex analysis
3. Linear algebra

These are important for intermediate and advanced courses in physics.
I will sometimes (but not always) use some standard notation. It is good to see because it sometimes appears in the supplementary references.

\( a \in A \) \( \quad a \) is a member of the set \( A \).

\( \forall \) "for all"

\( \exists \) "there exists"

\( \therefore \) "therefore"

\( \therefore \) "such that"

\( \Rightarrow \) implies

\( \Leftrightarrow \) if and only if

There are some sets that are used enough that they have special names:

\[ \mathbb{N} = \text{non negative integers} \quad \{0,1,2,\ldots\} \]
\[ \mathbb{Z} \text{ integers - both negative and positive} \]
\[ \{-\infty, -2, -1, 0, 1, 2, \ldots\} \]

\[ \mathbb{Q} \text{ rational numbers} \]
\[ x = \frac{m}{n} \quad m, n \in \mathbb{Z} \quad n \neq 0 \]

\[ \mathbb{IR} \text{ real numbers} \]

\[ \mathbb{C} \text{ complex numbers} \]
\[ z = x + iy \quad x, y \in \mathbb{IR} \quad i^2 = -1 \]

We will also use some elementary operations on sets.

\[ S = \text{set} \]

\[ a \in S \quad a \text{ is an element of } S \]

Examples of sets:

\[ \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{IR}, \mathbb{C} \]

\[ \{\text{animals}\} \]

\[ \mathcal{P} = \text{permutation of } N \text{ objects} \]
$N = 3 \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$

This is a set with 6 elements—permutations are important for studying the physics of identical particles.

2 partitions of N particles:

(N=3)

(\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 3 & 1 \end{pmatrix})

The set of partitions are used to label the possible final configurations in a scattering experiment.

(12)(3) means particle 1 and 2 bound together, and 3 is free.
Operations on sets

\[ S_1 \cup S_2 = S_1 + S_2 \]
\[ a \in S_1 \cup S_2 \] if \( a \in S_1 \) or \( a \in S_2 \)
\[ \{1, 2, 3\} \cup \{3, 4, 5\} = \{1, 2, 3, 4, 5\} \]

\[ S_1 \cap S_2 = a \in S_1 \cap S_2 \] if \( a \in S_1 \) and \( a \in S_2 \)
\[ \{1, 2, 3\} \cap \{3, 4, 5\} = \{3\} \]

\[ S_1 - S_2 = a \in S_1 \] and \( a \not\in S_2 \)
\[ \{1, 2, 3\} - \{3, 4, 5\} = \{1, 2\} \]
\[ \{3, 4, 5\} - \{1, 2, 3, 5\} = \{4, 5\} \]

\[ S_1 = S_2 \] \( a \in S_1 \) if and only if \( a \in S_2 \)

\[ S_1 \supseteq S_2 \] \( a \in S_1 \) if \( a \in S_2 \)
\[ S_2 \subseteq S_1 \]

This condition means that \( S_2 \) is a subset of \( S_1 \).
These properties can be used to establish properties of sets that may not be immediately obvious.

\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]

To show the equality of these sets, we need to show that if

\[ a \in A \cup (B \cap C) \Rightarrow a \in (A \cup B) \cap (A \cup C) \]

and

\[ a \in (A \cup B) \cap (A \cup C) \Rightarrow a \in A \cup (B \cap C) \]

(1) \( a \in A \cup (B \cap C) \Rightarrow \)

either \( a \in A \) or \( a \in B \) and \( C \)

\[ a \in A \cup B \quad \text{and} \quad a \in A \cup C \]

\[ a \in (A \cup B) \) and \( a \in (A \cup C) \]

\[ a \in (A \cup B) \cap (A \cup C) \]

(2) \( a \in (A \cup B) \cap (A \cup C) \)

\[ a \in A \cup B \quad \text{and} \quad a \in A \cup C \]

\[ a \in A = \text{true} \]

\[ a \in B, a \notin A \Rightarrow a \in C \]

\[ a \in C, a \notin A \Rightarrow a \in B \]
\[ a \in A \cup (B \cap C) \]

This shows that both sets are identical.

Sets come in different sizes that are relevant to problems in physics.

(A) Finite sets
(B) Countably infinite sets
(C) Uncountably infinite sets

A finite set has \( N \) elements if there is a 1-1 correspondence between the elements of the set and the integers \( \{1, \ldots, N\} \).

Examples: \# of permutations of three particles, \# of partitions of 4 particles...
A set is countably infinite if there is a 1-1 correspondence between elements of the set $S$ and elements of $\mathbb{N}$.

An infinite set is uncountable if it is not countable.

While there are different sizes of uncountable sets, but they do not appear to be important in physics applications.

Example: $\mathbb{Q}$ is countable

$$\mathbb{Q} = \{ (-1)^n \frac{m}{n} \mid l = 0 \text{ or } 1 \}$$

To construct the correspondence consider the subset of $\mathbb{N}$ given by

$$S = \{ 2^l 3^m 5^n \mid l = 0, 1; \quad n \neq 0 \}$$

These integers are ordered by size. The smallest one

$$1 \quad 2 \quad 3 \quad 5 \times 10 \times 15 \times \cdots$$

$$\begin{array}{ccc}
3^2 & 0 & 1 \\
5^3 & 0 & 0 \\
7^2 & 1 & 1
\end{array}$$
This gives the desired correspondence.
This works because every element of \( \mathbb{N} \) has a unique factorization as a product of prime numbers.

Not everything about counting is obvious.

* There is a rational number between any pair of real numbers.
* There is a real number between any pair of rational numbers.

But: The irrational numbers are not countable.

I will show that the volume of irrational numbers in \([0,1]\) is 1, while the volume of rational numbers in \([0,1]\) is 0.

* Consider rational numbers of the form \( \frac{n}{m} \) with \( m > 0 \) (\( m, n \in \mathbb{N} \)).
for each value of \( m \) there are \( m+1 \) possible values of \( n \).

Consider the volume

\[
\{ x \mid \frac{n}{m} - \frac{\varepsilon}{(m+1)^3} < x < \frac{n}{m} + \frac{\varepsilon}{(m+1)^3} \}
\]

we remove this volume for each \( n \) from 0...m. The total removed volume is

\[
V_m = (m+1) \times \frac{2\varepsilon}{(m+1)^3} = \frac{2\varepsilon}{(m+1)^2}
\]

The total volume for all \( m \) is

\[
V = \sum_{m=1}^{\infty} \frac{2\varepsilon}{(m+1)^2} = 2\varepsilon \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)
\]

\[
\frac{1}{2} \int_1^{\infty} \frac{1}{x^2} \, dx = 1
\]

\[
\leq 2\varepsilon \int_1^{\infty} \frac{1}{x^2} = 2\varepsilon
\]

Since \( \varepsilon \) is arbitrary we can make this volume as small as desired without eliminating any rational numbers \( \mathbb{Q} \). The volume of the rationals is 0. The volume of what remains in \( \mathbb{R} \)
what does this have to do with physics

classical 3 body problem

sun = infinite, massive
earth, jupiter

If we ignore the gravitational interaction between Earth and Jupiter then energy and angular momentum of the Earth and Jupiter are separately conserved. Each one orbits the sun with a different period.

What happens if we include the small interaction between Earth and Jupiter?

In this case only the total energy and angular momentum of the system are conserved. These conservation laws allow me earth to move off to 0,
with a small perturbation does the earth orbit the sun forever?

when the interaction is turned on the period of both planets depends on the initial conditions.

If the initial periods have sufficiently irrational ratios earth orbits the sun forever.

sufficiently irrational means that a continued fraction approximation converges slowly.

the conclusion — in the limit that the size of the interaction → 0 the probability of picking an initial condition with rational period ratios → 0

example 2 consider an infinite sequence of coin flips: HHHHHHHHH...

this is like an infinite decimal expansion of a point in [0,1] — in base 2, this is an uncountable set.
Distance measurements

In physics we work with real and complex numbers, vectors and functions. In computations it is important to know when two elements of the same set are close.

All of the distance functions that we will use share some basic properties:

\[ p(a, b) = \text{distance between } a, b \]

\[ p(a, b) \geq 0 \quad \text{the distance between any two elements is non-negative} \]

\[ p(a, b) = 0 \quad \text{if the distance between two elements is } 0, \text{ they are the same} \]

For any \( c \)

\[ p(a, b) \leq p(a, c) + p(c, b) \]

This is called the triangle inequality.
This simply says that the distance between 2 vertices in a triangle is less than the sum of the distances between the other 2 vertices.

distance functions with these 3 properties are called metrics.

sets with metrics are called metric spaces.

example: vectors

\[ p(\vec{v}_1, \vec{v}_2) = \sqrt{(\vec{v}_1 - \vec{v}_2) \cdot (\vec{v}_1 - \vec{v}_2)} \]

example: continuous functions on \([0,1]\)

\[ p_1(f_1, f_2) = \max_{x \in [0,1]} |f_1(x) - f_2(x)| \]

\[ p_2(f_1, f_2) = \int_0^1 |f_1(x) - f_2(x)| \, dx \]
Distance functions can be used to extend familiar concepts like continuity and convergence to more abstract settings.

Here we assume $S$ is a metric space with metric $\rho(a,b)$.

**Def:** A *Neighbourhood* of $P \in S$ is a set of the form

$$N_{P,R} = \{ a \in S | \rho(a,P) < R \}$$

where $R$ is a real positive number.

Note that the inequality $\rho(a,P) < R$ is strict, that is an important part of the definition.

**Def:** An *accumulation point* $P$ of $S' \subset S$ has the property that every neighbourhood of $P$ contains an element of $S'$.

$P$ does not have to be an element of $S'$. For example $0$ is an accumulation point of $(0,1) = \{ x \in \mathbb{R} | 0 < x < 1 \}$. 

Def: an interior point of $S' \subseteq S$ is a point $p \in S'$ with the property that there is a neighborhood of $p$ that is contained in $S'$.

- $0$ is not an interior point of $(0,1)$.
- $0.001$ is an interior point of $(0,1)$.

A set is open if every point of the set is an interior point.

$(0,1)$ is open.

A set is closed if it contains all of its accumulation points.

$[0,1]$ is closed.

The usual definition of continuity can be extended to functions $f$ that map metric space $A$ to metric space $B$.

$f: A \rightarrow B$ is continuous at $a$ if
for every \( \epsilon > 0 \) there is a neighborhood \( N_{a,R} \) with the property

\[
a' \in N_{a,R} \quad \Rightarrow (f(a') - f(c)) < \epsilon
\]

There is a stronger notion of continuity where the \( R \) does not depend on \( a \).

\( f(a) = b \) is uniformly continuous in \( A' \) if for every \( \epsilon > 0 \) there is an \( R \) (independent of \( a \in A' \)) such that in

\[
c \in N_{a,R} \quad \Rightarrow (f(c) - f(a)) < \epsilon
\]

Example

\( f(x) = \frac{1}{x} \) is continuous at any point in \((0, 1)\)

It is not continuous at 0, and is not uniformly continuous on any neighborhood of 0.
In this we will sometimes have to prove things.

We will use 3 general methods:

1. **Proof by construction**
2. **Proof by contradiction**
3. **Proof by induction**

**Examples**

**Proof by construction**

There is a prime number \( > 11 \)

\[ 13 > 11 \]

\( 13 \) \( \div n \) is not an integer for \( n = 2, 3, \ldots, 12 \).

\therefore 13 is a prime \( \# > 11 \)

**In quantum field theory the goal is to construct a theory satisfying expected properties**

1. The theory is a quantum theory.
2. The theory is consistent with special relativity.
3. The energy of the system is bounded from below (i.e., the theory is stable).
(4) The theory describes particles.
(5) The ground state is unique.
(6) Experiments performed in spacelike separated regions are independent.

So far there are no known non-trivial examples of theories satisfying all of these conditions. The question is: are the requirements logically consistent?

The answer is yes because they are satisfied by non-interacting quantum field theories.

If they were inconsistent then physicists would be wasting their time looking for solutions to this problem.

Proof by contradiction:

Example: Aristotle $\sqrt{2}$ is irrational?
by contradiction assume

\[ \sqrt{2} = \frac{m}{n} \]

with no common divisors (if they were there they could be divided out)

squaring

\[ 2n^2 = m^2 \]

since \(2n^2\) is even \(m\) must be even (the square of an odd \# must be odd because it has no factors of 2)

\[ m^2 \text{ is divisible by 4} \]

\[ n^2 = \frac{m^2}{2} \text{ is even so } n \text{ must also be divisible by 2} \]

\[ \therefore 2 \text{ is a common divisor of both } m \text{ and } n \text{. This contradicts the assumption that } \sqrt{2} \text{ is rational} \]
proof by induction - this is used a lot in physics applications
the procedure is
- show something holds for n=1
- show that if it holds for n then it holds for n+1

example - creation and annihilation operators in quantum theory

In quantum theory
\[ [x, p] = i\hbar \]
define \( a \) and \( a^\dagger \) by
\[ x = \sqrt{\hbar} (a + a^\dagger) \quad p = i\sqrt{\hbar} (a^\dagger - a) \]
solving for \( a, a^\dagger \) in terms of \( x, p \)
\[ a = \sqrt{\frac{\hbar}{i}} (x + ip) \quad a^\dagger = \sqrt{\frac{\hbar}{i}} (x - ip) \]
problem - show \( [a, (a^\dagger)^n] = n(a^\dagger)^{n-1} \)
a and \( a^\dagger \) are called creation and annihilation operators

\[
\begin{align*}
\{a, a^\dagger\} &= \frac{1}{2\hbar} \{x + i\hbar, x - i\hbar\} \\
&= \frac{1}{2\hbar} (\{x, x\} + i\{p, x\} - i\{x, p\} + \{p, p\}) \\
&= 1
\end{align*}
\]

Assume

\[
\{a, (a^\dagger)^n\} = n(a^\dagger)^{n-1}
\]

\[
\{a, (a^\dagger)^{n+1}\} = a(a^\dagger)^n - (a^\dagger)^{n+1} a =
\]

\[
a a^\dagger (a^\dagger)^n - a^\dagger a (a^\dagger)^n + a^\dagger a (a^\dagger)^n - (a^\dagger)^{n+1} a
\]

\[
\begin{align*}
\{a, a^\dagger\} (a^\dagger)^n + a^\dagger \{a, (a^\dagger)^n\} \\
&= 1 - n(a^\dagger)^{n-1} \quad \text{(by induction assumption)}
\end{align*}
\]

\[
(a^\dagger)^n (1 + n) = (n+1)(a^\dagger)^n
\]

This shows the result holds for \( n \) if it holds for \( n+1 \).