Lecture 10

The residue theorem

\[ \oint_C f(z) \, dz = 2\pi i \sum \text{Res } f(z_i) \]

"z_i" are the isolated singular points in the interior of the curve \( C \), where \( C \) is oriented in the counterclockwise direction.

If \( C \) is in the clockwise direction this becomes

\[ \oint_C f(z) \, dz = -2\pi i \sum \text{Res } f(z_i) \]

The curve must be in a region where \( f(z) \) is analytic.

\[ \int_R f(z) \, dz = 2\pi i \sum_{j=1}^{3} \text{Res } f(z_j) \]

The result follows from Cauchy's theorem.
\[ \oint_{\gamma} f(z) \, dz = 0 \quad \text{and} \quad 0 = \oint_{\gamma} f(z) \, dz - \sum_{i} \oint_{\gamma_i} f(z) \, dz \]

where by definition

\[ \text{Res } f(z) = \frac{1}{2\pi i} \oint_{\gamma} f(z) \, dz \]

where \(\gamma_i\) is a counterclockwise curve in \(\mathcal{R}\) containing only 1 singular point.

To compute the residue note that \(f(z)\) has a convergent Laurent series about any singular point

\[ f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z - z_0)^n} \]

To compute the residue

\[ \oint_{\gamma} f(z) = \sum_{n=0}^{\infty} a_n \oint_{\gamma} (z - z_0)^n + \sum_{n=1}^{\infty} b_n \oint_{\gamma} \frac{dz}{(z - z_0)^n} \]

vanishes by \(Z - Z_0 = re^{i\theta}\)

Cauchy's Thm \(dz = ire^{i\theta} d\theta\)
\[
\ln \int_0^{2\pi} \frac{ire^{i\phi}}{r^n e^{i(n-1)\phi}} \, d\phi = \ln \frac{1}{\pi r^{n-1}} \int_0^{2\pi} e^{-i(n-1)\phi} \, d\phi
\]

This vanishes for any sufficiently small \( r \) because \( e^{-i(n-1)\phi} \) is periodic unless \( n = 1 \), then it becomes 2\(\pi i b_1 \).

\[
\therefore \text{Res} (f(z)) = b_1 = \frac{1}{2\pi i} \int \frac{f(z)}{z} \, dz
\]

where \( b_1 \) is the coefficient of \( \frac{1}{z-z_0} \) in the Laurent series.

If \( f(z) \) has a pole of order \( n \) at \( z_0 \), then

\[
f(z) = \frac{g(z)}{(z-z_0)^n}
\]

where \( g(z) \neq 0 \), \( g(z) \) analytic at \( z_0 \).

To find \( b_1 \), note

\[
g(z) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m g}{dz^m} (z_0) (z-z_0)^m
\]

In this case,

\[
b_1 = \frac{1}{(n-1)!} \frac{d^{n-1} g}{dz^{n-1}} (z_0) \text{ pole of order } n
Note that even if \( f(z) \) has an isolated essential singularity it still has a convergent Laurent series

\[
\text{Res } (f(z)) = b_1
\]

because the series can be integrated term by term due to the convergence.

**Example 1**

Let \( g(z) \) be analytic in a region \( R \) with \( g(z_0) \neq 0 \). Define

\[
f(z) = \frac{g(z)}{(z-z_0)}
\]

\( f(z) \) has a pole of order 1 at \( z = z_0 \). By inspection \( b_1 \) is

\( g(z_0) \neq 0 \)

\[
\oint_C f(z) \, dz = 2\pi i \cdot g(z_0)
\]

for any contour clockwise path in \( R \) that has \( z_0 \) in the interior.
example 2

Let \( g(z) \) be analytic in \( R \)
and define

\[ f(z) = \frac{g(z)}{(z-z_0)^n} \quad z \in R \]

By analyticity Taylor’s theorem give

\[ f(z) = \sum_{k=0}^{\infty} \frac{g^{(k)}(z_0)}{k!} \frac{1}{(z-z_0)^{n+k}} \]

\[ + \frac{1}{(n-1)!} \frac{d^{n-1}g}{d(z-z_0)^{n-1}}(z_0) \frac{1}{(z-z_0)^{n+1}} + \text{analytic terms} \]

\[ \text{Res} \ f(z) \bigg|_{z=z_0} = \frac{1}{(n-1)!} \frac{d^{n-1}g}{d(z-z_0)^{n-1}}(z_0) \]

\[ \int f(z) \, dz = \frac{2\pi i}{(n-1)!} \frac{d^{n-1}g}{d(z-z_0)^{n-1}}(z_0) \]

example 3

\[ \int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \int_{-\infty}^{\infty} \frac{dv}{(x+ia)(x-ia)} \]

Consider instead

\[ \oint \frac{dz}{z^2+a^2} \quad \gamma = \]

\[ \begin{array}{c}
\end{array} \]
The integral has the form

\[ \oint \frac{dz}{(z-i\alpha)(z+i\alpha)} = 2\pi i \frac{1}{i\alpha - i\alpha} \]

residue

\[ \frac{\Pi}{\alpha} = \int_{-R}^{R} \frac{dx}{x^2 + a^2} + \int_{0}^{\pi} \frac{i R \Re^i \phi \, d\phi}{(R^2 \Re^2 i \phi + a^2)} \]

\[ = \int_{-R}^{R} \frac{dx}{x^2 + a^2} + \left( \frac{i}{R} \int_{0}^{\pi} \frac{e^{i \phi}}{e^{2 i \phi} + a^2 R^2} \, d\phi \right) \]

\[ = \int_{-R}^{R} \frac{dx}{x^2 + a^2} + \left( \frac{i}{R} \int_{0}^{\pi} \frac{d\phi}{e^{i \phi} + \frac{a^2}{R^2} e^{-i \phi}} \right) \]

Note

\[ |e^{i \phi} + \frac{a^2}{R^2} e^{-i \phi}| = \sqrt{1 + \frac{a^2}{R^2} + 2 \frac{a^2}{R^2} \cos \phi} \leq 1 - \frac{a^2}{R^2} \]

The second integral is bounded by

\[ \frac{\Pi}{1 - a^2 / R^2} \]

Taking the limit \( R \to \alpha \)

\[ \frac{\Pi}{\alpha} = \int_{-\infty}^{\infty} \frac{dx}{x^2 + a^2} + i \lim_{R \to \infty} \frac{1}{R} \int_{-R}^{R} \Re f(R) \, d(R) \quad |f(R)| \leq \frac{\Pi}{1 - a^2 / R^2} \]

\[ \int_{-\infty}^{\infty} \frac{dx}{x^2 + a^2} = \frac{\Pi}{\alpha} \]

\[ \int_{-\infty}^{\infty} \frac{dx}{x^2 + a^2} = \frac{\Pi}{\alpha} \]
If we closed the contour in the lower half plane

\[ \gamma = - \gamma \]

as before - as \( R \to \infty \) the contribution from the semicircle vanishes and what remains is

\[ \int_{-\infty}^{\infty} \frac{dx}{x^2 + a^2} = \frac{\pi}{a} \]

example 4

\[ \int_{-\infty}^{\infty} e^{iax^2} \, dx = 2 \int_{0}^{\infty} e^{iax^2} \, dx \]

Consider

\[ \oint_C e^{iaz^2} \, dz \]

the integral around \( \gamma \) vanishes by Cauchy's theorem because \( e^{iaz^2} \) is entire.
For the path

\[ Z = \gamma \]

\[ Z = R e^{i\phi} \]

\[ Z = \lambda e^{i\phi} \]

\[ 0 = \int_0^R e^{-\alpha x^2} d\alpha + i R \int_0^{\pi/4} e^{i\phi} d\phi \]

\[ \int_0^R e^{-\alpha x^2} d\alpha = \sqrt{i} \int_0^R -\alpha^{1/2} d\alpha + i R \int_0^{\pi/4} e^{i\phi} d\phi \]

because \( R e^{-\alpha r^2 \sin \theta} \to 0 \) as \( R \to \infty \) for \( 0 < \theta < \pi \).

Taking the limit \( R \to \infty \)

\[ \int_0^\infty e^{i\alpha x^2} dx = \sqrt{\frac{\pi}{i\alpha}} e^{-\frac{1}{4\alpha}} \]

\[ \int_0^\infty e^{-\alpha x^2} dx = \frac{\sqrt{\pi}}{2} \int_0^\infty e^{-\alpha x^2} dx \]
The integral
\[ \int_{-\infty}^{\infty} e^{-\frac{x^2}{a}} \, dx = (\text{let } u = \sqrt{a} x) \]
\[ \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{\sqrt{a}} = \frac{1}{\sqrt{a}} \, I \]

\[ I = \int_{-\infty}^{\infty} e^{-u^2} \, du \]

\[ I^2 = \int_{-\infty}^{\infty} e^{-x^2} \, dx \cdot \int_{-\infty}^{\infty} e^{-y^2} \, dy = \pi \int_{0}^{\infty} e^{-r^2} \, r \, dr \]

\[ \pi \int_{0}^{\infty} e^{-r^2} \, r \, dr = \frac{\pi}{2} \int_{0}^{\infty} e^{-r^2} \, 2r \, dr = \pi \]

\[ \pi \int_{0}^{\infty} e^{-v} \, dv = -\pi e^{-v} \bigg|_{0}^{\infty} = \pi \]

\[ \therefore \quad I = \sqrt{\pi} \]

\[ \int_{-\infty}^{\infty} e^{i\alpha x^2} \, dx = \sqrt{\frac{i\pi}{a}} \]

Example 5
\[ \int_{0}^{\infty} \frac{\sin x}{x} \, dx = \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} \frac{\sin x}{x} \, dx \]

Note - at this point the \( \epsilon \) does nothing because
\[ \frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \, x^{2n} \]

is entire - and well behaved at \( x = 0 \).
however we can write this as

\[ \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \left( \frac{e^{ix}}{2i(x-i\epsilon)} - \frac{e^{ix}}{2i(x-i\epsilon)} \right) \]

in this form each term has a pole at \( x = i\epsilon \)

consider the following 2 contours

\[ \oint_{C_1} \frac{e^{iz}}{2i(z-i\epsilon)} \quad \oint_{C_2} \frac{e^{-i\epsilon}}{2i(z-i\epsilon)} \]

\[ \oint_{C_1} \frac{e^{iz}}{2i(z-i\epsilon)} = (2\pi i), \quad \frac{e^{-i\epsilon}}{2i} \to \pi \text{ as } \epsilon \to 0 \]

\[ \oint_{C_2} \frac{e^{-i\epsilon}}{2i(z-i\epsilon)} = 0 \text{ by Cauchy's theorem} \]

note

\[ \oint_{C_1} \frac{e^{iz}}{2i(z-i\epsilon)} \, dz = \]

\[ \int_{-R}^{R} \frac{e^{ix}}{2i(x-i\epsilon)} \, dx + \int_{0}^{\pi} \frac{e^{iR \cos \phi + i\sin \phi}}{2i(R \cos \phi - i\epsilon)} \, iR \, d\phi \]
\[ \int_{-R}^{R} \frac{e^{ix}}{2i(x-i\epsilon)} + \int_{0}^{\pi} e^{-R \sin \phi} \left( \cos (R \cos \phi) + i \sin (R \cos \phi) \right) d\phi \]

The second integral vanishes as \( R \to \infty \) - this is because it vanishes for any \( \phi \neq 0 \) or \( \pi \) and the contribution to the integral near 0 or \( \pi \) vanishes as \( d\phi \to 0 \).

Taking the limit \( R \to \infty \) gives

\[ \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx = \lim_{\epsilon \to 0} \int_{-R}^{R} \frac{e^{ix}}{2i(x-i\epsilon)} = \pi \]

Note that the integral in example 4 appears in Feynman's path integral, while the integral in example 5 appears in partial wave quantum scattering.

The observation that

\[ \lim_{R \to \infty} \int_{0}^{\pi} e^{-R \sin \phi} d\phi = 0 \]

is called Jordan's lemma.
example 6

\[
\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^2} = \int_{-\infty}^{\infty} \left(-\frac{1}{2a} \frac{d}{da}\right) \frac{1}{x^2 + a^2} \, dx
\]

\[
= -\frac{1}{2a} \frac{d}{da} \left(\int_{-\infty}^{\infty} \frac{1}{x^2 + a^2} \, dx\right)
\]

integral from example 1

\[
= -\frac{1}{2a} \frac{d}{da} \left(\frac{\pi}{a}\right) = \frac{\pi}{2a^3}
\]

example 7

\[
\int_{-\infty}^{\infty} e^{iax - bx^2} \, dx
\]

in this case the first step is to complete the square

\[-bx^2 + iax = -b(x - \frac{ia}{2b})^2 - \frac{a^2}{4b}\]

the integral becomes

\[
\int_{-\infty}^{\infty} e^{-b(x - \frac{ia}{2b})^2 - \frac{a^2}{4b}} \, dx
\]

consider

\[
-\frac{a^2}{4b} \int_{-\infty}^{\infty} e^{-bz^2} \, dz
\]
Using the contour
\[ z = x \quad d\overline{z} = dx \]

\[ z = -R - iy \quad d\overline{z} = -idy \]

\[ z = x - i\frac{a}{2b} \quad d\overline{z} = dx \]

This integral has 4 parts.

The integral around the closed curve is 0 by Cauchy's theorem since the integrand is entire.

\[ 0 = e^{-\frac{a^2}{4b}} \oint e^{-b\overline{z}^2} d\overline{z} = \]

\[ e^{-\frac{a^2}{4b}} \left( \int_{-R}^{R} e^{-bx^2} \, dx + \int_{0}^{\frac{a}{2b}} e^{-b(R^2 - 2iyR + y^2)} \, dy \right) \]

\[ + \int_{R}^{0} e^{-b(x - i\frac{a}{2b})^2} \, dx + \int_{0}^{\frac{a}{2b}} e^{-b(R^2 + 2iyR + y^2)} \, dy \right) \]

The second and 4th integrals are bounded.

\[ |II| < e^{-R^2 \frac{a}{2b}} \]

which vanishes as \( R \to \infty \).
taking the limit $R \to \infty$ what remains is

$$\frac{a^2}{4b} \int_{-\infty}^{\infty} e^{-bx^2} dx = \frac{a^2}{4b} \int_{-\infty}^{\infty} e^{-b(x-i \frac{a}{2b})} dx$$

the integral on the right is

$$\int_{-\infty}^{\infty} e^{i ax - bx^2} dx$$

while the integral on the left is

$$\frac{a^2}{4b} \sqrt{\frac{\pi}{b}}$$

$$\therefore \int_{-\infty}^{\infty} e^{i ax - bx^2} = \sqrt{\frac{\pi}{b}} e^{-\frac{a^2}{4b}}$$

example 8

$$\int_{\gamma} \frac{f(z) \, dz}{(z-a_1) \cdots (z-a_n)} = \int_{\gamma} g(z) \, dz$$

where $f(z)$ is analytic and not 0 at $a_1$, $a_2$, and all of the $a_i$'s are different.
The integrand has poles of order 1 at $a_i$, $n$. Assume that all of the poles are in the interior of $\gamma$.

Note that the residue at $a_i$ is $b_i$ in the Laurent series

$$\text{Res} \left( g(a_i) \right) = \frac{f(a_i)}{(a_i-a_1) \cdots (a_i-a_{i-1})(a_i-a_{i+1}) \cdots (a_i-a_n)}$$

$$= \frac{f(a_i)}{i\pi (a_i-a_j)}$$

It follows immediately that

$$\oint g(z) dz = 2\pi i \sum_{i=1}^{n} \frac{f(a_i)}{i\pi (a_i-a_j)}$$

If $\gamma$ does not enclose all of the poles then the sum is just over the poles in the interior of $\gamma$. 
example 10

The residue theorem can also be used to sum infinite series.

Theorem. Let $f(z)$ be a meromorphic function and let $\gamma$ be a curve that

1. encloses the zeroes of $\sin(\pi z)$;
2. assume that the poles of $f(z)$ are distinct from the $0$s of $\sin(\pi z)$.

(The $0$s of $\sin(\pi z)$ are $-N, -N-1, \ldots, 0, 1, 2, \ldots, N$.)

The poles and residues of $\pi \cot(\pi z)$

$$\pi \frac{\cos(\pi z)}{\sin(\pi z)} = 0$$

Expanding $\sin(\pi z)$ about $n$

$$0 + (z-n) \pi \cos(\pi n) + o((z-n)^2)$$

The residue of $\pi \cot(\pi z)$ at $n$ is

$$\frac{\pi \cos(\pi n)}{\pi \cos(\pi n)} = 1$$
\[
\left| \cot \left( \pi z \right) \right| = \left| \frac{\cos \left( \pi (x + iy) \right)}{\sin \left( \pi (x + iy) \right)} \right| = \\
\left| \frac{\cos (\pi x) \cosh (\pi y) - i \sinh (\pi x) \sin (\pi y)}{\sin (\pi x) \cosh (\pi y) + i \cos (\pi x) \sinh (\pi y)} \right| \\
\left| \frac{\cos^2 (\pi x) \cosh^2 (\pi y) + \sin^2 (\pi x) \sinh^2 (\pi y)}{\sin^4 (\pi x) \cosh^2 (\pi y) + \cosh^2 (\pi x) \sinh^4 (\pi y)} \right|^\frac{1}{2}
\]

This is bounded for large \( y \);

when \( y = 0 \) it is bounded in \( x = n + \frac{1}{2} \)

(when \( \sin \pi x = 1 \))

Note that if \( f(z) \) vanishes faster than \( 1/|z| \) for large \( |z| \) and we consider large rectangles.

The integral

\[
\frac{1}{2\pi i} \oint_C \pi \cot (\pi z) f(z) \, dz \to 0
\]
In the limit of an infinite rectangle that crosses the $x$ axis at $n+\frac{1}{2}$

$$0 = \sum_{n} f(n) + \sum_{\text{poles of } f(z)} \text{Res } \left( \frac{\pi \cot(\pi z)}{f(z)} \right)$$

**Example**

Let $P(z) = \prod_{i=1}^{n} (z-a_i)$, $a_i \neq n$.

\[
\sum_{m=-\infty}^{\infty} \frac{1}{\pi (m-a_i)} = -\sum_{i=1}^{n} \frac{\pi \cot(\pi a_i)}{\pi (a_i-a_j)}
\]

**Example**

\[
\sum_{m=-\infty}^{\infty} \frac{1}{m^2 + a^2} = -\left( \frac{\pi \cot(i\pi a)}{2ia} - \frac{\pi \cot(-i\pi a)}{-2ia} \right)
\]

\[
= -\frac{i}{\pi a} \cdot \frac{\cosh(\pi a)}{\sinh(\pi a)} = \frac{\cosh(\pi a)}{a} = \coth(\pi a)
\]
note that to use this result we need to be sure that the function \( g(z) \) falls off fast enough at \( \infty \).

**Isolated essential singularities**

If \( g(z) \) has an isolated essential singularity then

\[
g(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}
\]

is a convergent series for \( z \) between 2 circles about \( z_0 \). If \( g(z) \) has no other singularities between the circles

\[
\oint_Y g(z) \, dz = 2\pi i \, b_1,
\]

which follows by term by term integration in the convergent series.
So far we have considered the cases where the pole is inside of \( \gamma \) or outside of \( \gamma \). What happens if the pole is on the curve?

Consider

\[
\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0} \, dx
\]

where \( f(z) \) is analytic and non 0 at \( x = x_0 \).

Strictly speaking, this integral is not well defined. To make it well defined we will disturb the contour.

The different choices will give different results.
The integral is the sum of 3 integrals

\[ \int_{-\infty}^{\infty} \frac{f(z)}{z-x_0} \, dz = \int_{-\infty}^{x_0 \sim} \frac{f(y)}{x-x_0} + \int_{x_0 \sim}^{\infty} \frac{f(z)}{z-x_0} + \int_{x_0 \sim}^{\infty} \frac{f(x)}{x-x_0} \]

The middle integral can be done using

\[ z - x_0 = \varepsilon e^{i\phi} \]

\[ d\varepsilon = i\varepsilon e^{i\phi} \, d\phi \]

\[ \int \frac{f(x_0 + \varepsilon e^{i\phi})}{\varepsilon e^{i\phi}} \, i\varepsilon e^{i\phi} \, d\phi = i f(x_0) \]

\[ = \frac{1}{\pi} f(x_0) \]

The sum of the other 2 terms is called the principal value

\[ P \int_{-\infty}^{\infty} f(x) \, dx = \lim_{\epsilon \to 0} \left( \int_{-\infty}^{x_0 - \epsilon} \frac{f(x)}{x-x_0} + \int_{x_0 + \epsilon}^{\infty} \frac{f(x)}{x-x_0} \right) \]

To compute the principal value (or at least show that it is finite.)
If \( f(x) \) is analytic near \( x_0 \),

\[
P \lim_{\epsilon \to 0} \frac{f(z)}{z-x_0} = \lim_{\epsilon \to 0} \left[ \int_{-\infty}^{\epsilon} \frac{f(x)}{x-x_0} \, dx + \int_{\epsilon}^{x_0-\epsilon} \frac{f(x)}{x-x_0} \, dx + \int_{x_0+\epsilon}^{0} \frac{f(x)}{x-x_0} \, dx + \int_{0}^{b} \frac{f(x)}{x-x_0} \, dx \right]
\]

we assume that \( f(x) \) has a convergent taylor series on \( a, b \)

\[
f(x) = f(x_0) + \sum_{n=1}^{\infty} \frac{(x-x_0)^n}{n!} a_n
\]

then the middle 2 integrals become

\[
f(x) \int_{x_0}^{x_0-\epsilon} \frac{dx}{x-x_0} + f(x) \int_{x_0+\epsilon}^{b} \frac{dx}{x-x_0} + \\
\int_{a}^{b} \sum_{n=1}^{\infty} \frac{(x-x_0)^n}{n!} a_n \, dx
\]

\[
\int_{a}^{b} \sum_{n=1}^{\infty} \frac{1}{n} \frac{(x-x_0)^n}{n!} a_n \, dx
\]

the log term becomes \(-\epsilon \) and \( a-x \)

are both negative

\[
f(x_0) \ln \frac{b-x_0}{x_0-a} + \sum_{n=1}^{\infty} \frac{1}{n!} a_n (b-x_0)^n - (a-x_0)^n
\]
both terms are finite - the other terms
\[ \int_{-\infty}^{a} \frac{f(x)}{x-x_0} \, dx + \int_{b}^{\infty} \frac{f(x)}{x-x_0} \, dx \]
will be finite if \( f(x) \) is well behaved at \( x \to \pm \infty \). This shows that the principal value is generally finite

what happens when the limit is taken - there are 2
\( \ln \) singularities that cancel
\[ \ln e - \ln (x-a) + \ln (b-x) - \ln e \]

\[ \int \frac{f(x)}{x-x_0} \, dx = P \int \frac{f(x)}{x-x_0} + \text{Im} f(x_0) \]
\[ \text{depends on how to treat the pole} \]

* In applications the choice of \( \ln e \) is not arbitrary - in scattering problems it distinguishes between incoming and outgoing waves
\[ H = H_0 + V \]

\[(H_0 - E)\Psi = 0 \]

\[(H_0 - E)\Psi = -V\Psi \]

\[\Psi = \Psi_0 + (E - H_0)^{-1} V\Psi \]

\[E \rightarrow E \pm \epsilon \text{ determines } \Psi, \text{ this is a solution that moves} \]