Lecture 11

Using the residue theorem

\[ \oint f(z) \, dz = 2\pi i \sum \text{Res } f(z, z_i) \]

Definitions

\( \gamma \) is a closed piecewise continuous counterclockwise path in the complex plane.

\( z_i \) are the isolate poles in the region enclosed by \( \gamma \) (not on the boundary about each isolated singularity

\[ f(z) = \sum_{n=0}^{\infty} a_n (z-z_i)^n + \sum_{n=1}^{\infty} b_n (z-z_i)^{-n} \]

\( \text{Res } f(z_i) = b_1 \) in the above expansion

If \( \gamma \) is clockwise the right side of the above is multiplied by \(-1\)

Last time applications

1. \[ \oint \frac{g(z)}{z-z_i} \, dz = 2\pi i \, g(z_i) \quad g(z_i) \neq 0, \quad g(z) \text{ analytic in region enclosed by } \gamma \]

2. \[ \oint \frac{g(z)}{(z-z_i)^n} \, dz = 2\pi i \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} g(z_i) \quad \frac{d^n}{dz^n} g(z_i) \neq 0, \quad g(z) \text{ analytic in region enclosed by } \gamma \]
\[ \int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \frac{\pi}{a} \]

\[ \int_{-\infty}^{\infty} e^{iax^2} \, dx = \sqrt{\frac{\pi}{a}} \]

\[ \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx = \pi \]

\[ \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3} \]

\[ \int_{-\infty}^{\infty} e^{-bx^2+iax} \, dx = e^{-\frac{a^2}{4b}} \sqrt{\frac{\pi}{b}} \]

Today we consider a few more examples.

6. Let \( f(z) = \frac{g(z)}{\prod_{i=1}^{r} (z-z_i)} \)

Let \( \gamma \) encircle all of the poles of \( f(z) \), assume \( g(z_1) \neq 0 \) where \( g(z) \) is analytic in the interior of \( \gamma \)

\[ \oint f(z) \, dz = 2\pi i \sum_{n=1}^{r} \frac{g(z_n)}{\prod_{i \neq n} (z_n - z_i)} \]

This is an immediate consequence of the residue theorem.
The residue theorem can also be used to sum infinite series.

First consider

$$\pi \cot(\pi z) = \pi \frac{\cos(\pi z)}{\sin(\pi z)}$$

This function has simple poles at $z = n$, $n = \text{integers}$ between $-\infty \to \infty$.

To find the residue at the poles note

$$\sin(\pi z) = \sin - (Z-n) \frac{d\sin(\pi z)}{dz} (n) + \cdots$$

$$= (z-n) \pi \cos(\pi n) + O(z-n)^2$$

$$\pi \cot(\pi z) = \frac{\pi \cos(\pi n)}{\pi \cos(\pi n) (z-n) + \cdots}$$

This shows that the residue at each pole is 1.

To perform integration, we investigate the asymptotic properties of this function

$$\pi \cot (\pi z) = \pi \frac{\cos(\pi (x+iy))}{\sin(\pi (x+iy))} =$$

$$\pi \frac{\cos(\pi x) \cosh(\pi y) - i \sin \pi x \sinh \pi y}{\sin(\pi x) \cosh(\pi y) + i \cos(\pi x) \sinh(\pi y)}$$

$$\left| \pi \cot (\pi z) \right| = \pi \left| \frac{\cos^2(\pi x) \cosh^2(\pi y) + \sin^2(\pi x) \sinh^2(\pi y)}{\sin^2(\pi x) \cosh^2(\pi y) + \cos^2(\pi x) \sinh^2(\pi y)} \right|^{1/2}$$
for large $y$ \[ \sinh \cosh \to e^{\frac{\pi y}{2}} \]
\[ \begin{align*}
\pi |\cos (\pi x)| &\to \pi \left| \frac{\cos^2 (\pi x) + \sin^2 (\pi x)}{\sin^2 (\pi x) + \cos^2 (\pi x)} \right|^{\frac{1}{2}} \to 0 < \infty
\end{align*} \]
when $y = 0$
\[ \pi |\cos (\pi x)| \to \pi \left| \frac{\cos \pi x}{\sin \pi x} \right| \]
it is bounded if $x = n + \frac{1}{2}$

consider a curve represented by a large rectangle that has verticle at $x = \pm n + \frac{1}{2}$ for large $n$

Let $f(z)$ be another meromorphic function that vanishes faster than $\frac{1}{|z|}$ for large $z$

then the integral around a sufficiently large rectangle
\[ \int f(z) \pi \cot (\pi z) \, dz \to 0 \]

This is a consequence of the asymptotic properties of $f(z)$ and $\pi \cot (\pi z)$. 
using the residue theorem

\[
0 = 2\pi i \sum_{n=-\infty}^{\infty} \text{Res} f(z_i) \cdot \pi \cot(\pi z_i) + 2\pi i \sum_{n=-\infty}^{\infty} f(n)
\]

where we assume \[z_i \neq n\]

\[
\sum_{n=-\infty}^{\infty} f(n) = -\sum_{n=-\infty}^{\infty} \text{Res} f(z_i) \pi \cot (\pi z_i)
\]

example

\[
\sum_{m=-\infty}^{\infty} \frac{1}{\pi (m-a_i)} = -\sum_{i=1}^{\infty} \frac{1}{\pi (a_i-a_j)} \pi \cot (\pi a_i)
\]

specific case

\[
\sum_{m=0}^{\infty} \frac{1}{m^2+a^2} = \frac{1}{2} \sum_{m=-\infty}^{\infty} \frac{1}{m^2+a^2} + \frac{1}{2a^2} =
\]

\[
\frac{1}{2a^2} + \frac{1}{2} \cdot \left( -\frac{1}{2ia} \pi \cot (i\pi a) - \frac{1}{-2ia} \pi \cot (-i\pi a) \right)
\]

\[
\frac{1}{2a^2} + \frac{1}{2} \left( \frac{\pi \cosh (\pi a)}{2a \sinh (\pi a)} + \frac{\pi \cosh (\pi a)}{2a \sinh (\pi a)} \right)
\]

\[
\frac{1}{2a^2} + \frac{\pi}{2a} \coth (\pi a)
\]
we discussed the case that the singular points of $f(z)$ are inside or outside of the region bounded by $\gamma$.

Next we consider the case where the singularity lies on the curve

$\gamma$

consider

$$\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0} \, dx$$

where $f(z)$ is analytic and non zero at $x = x_0$.

Strictly speaking this integral is not well defined. To treat this we distort the curve

$\gamma$

or

we will see that the different choices will give different results
Integrals of this type appear in scattering theory, and the choice of how to distort the contour has physical consequences.

For the example it is useful to break the integral up into 3 parts:

\[
\int_{-\infty}^{\infty} \frac{f(z)}{z-z_0} \, dz = \int_{-\infty}^{x_0-\epsilon} \frac{f(x)}{x-x_0} \, dx + \int_{x_0}^{x_0+\epsilon e^{i\phi}} \frac{f(x+\epsilon e^{i\phi})}{e^{i\phi}} \, d\phi
\]

\[
+ \int_{x_0+\epsilon}^{\infty} \frac{f(x)}{x-x_0} \, dx
\]

The second integral as a limit as \( \epsilon \to 0 \) it is equal to

\[
f(x) \left( \mp i\pi \right) \quad \frac{-\epsilon}{u}
\]

The other 2 integrals cannot be done analytically in general.

We can show that it is finite provided \( f(x) \) falls off fast enough for large \( |x| \).
The remaining integral is called the principal value

\[ P \int_{-\infty}^{\infty} \frac{f(x)}{x-x_0} \, dx = \lim_{\varepsilon \to 0} \left( \int_{-\infty}^{x_0-\varepsilon} + \int_{x_0+\varepsilon}^{\infty} \right) \frac{f(x)}{x-x_0} \, dx \]

To understand this we write the term in (7) as

\[
\left( \int_{-\infty}^{x_0-b} + \int_{x_0-b}^{x_0-\varepsilon} + \int_{x_0+\varepsilon}^{x_0+b} + \int_{x_0+b}^{\infty} \right) \frac{f(x)}{x-x_0} \, dx
\]

where we assume \( b \) is small enough so \( f(x) \) has a convergent power series for \( |x-x_0|<b \)

\[ P \int_{-\infty}^{\infty} \frac{f(x)}{x-x_0} \, dx = \int_{-\infty}^{x_0-b} \frac{f(x)}{x-x_0} \, dx + \int_{x_0+b}^{\infty} \frac{f(x)}{x-x_0} \, dx \]

\[ \lim_{\varepsilon \to 0} \left[ \int_{x_0-b}^{x_0-\varepsilon} \frac{1}{x-x_0} \left( \sum_{n=1}^{\infty} a_n (x-x_0)^n \right) \, dx + \int_{x_0+\varepsilon}^{x_0+b} \frac{1}{x-x_0} \left( \sum_{n=1}^{\infty} a_n (x-x_0)^n \right) \, dx \right] 
\]

\[ = \left( \int_{-\infty}^{x_0-b} + \int_{x_0+b}^{\infty} \right) \frac{f(x)}{x-x_0} \]

\[ \sum_{n=1}^{\infty} \frac{a_n}{n+1} \left[ \left( -\varepsilon \right)^{n+1} - \left( -b \right)^{n+1} \right] + \sum_{n=1}^{\infty} \frac{a_n}{n+1} \left( b^{n+1} - \left( -\varepsilon \right)^{n+1} \right) \]

\[ a_0 \left( \ln \frac{-\varepsilon}{-b} + \ln \frac{b}{\varepsilon} \right) \]
Now the \( \lim_{\epsilon \to 0} \) can be taken

\[
P \int_{-\infty}^{\infty} \frac{f(x)}{x-x_0} \, dx = \\
\left( \int_{-\infty}^{x_0-\epsilon} + \int_{x_0+\epsilon}^{\infty} \right) \frac{f(x)}{x-x_0} + \sum_{n=1}^{\infty} \frac{a_n}{n!} \left( b^n - (-b)^n \right)
\]

where the series converges because \( f(x) \) is analytic on \( (x_0-\epsilon, x_0+\epsilon) \).

One example where this comes up is

\[
\int_{x_0-\epsilon}^{x_0+\epsilon} \frac{dx}{x^2 - x_0^2 + \epsilon}
\]

Here there are 2 singularities \( x = x_0 \pm \epsilon \).

\[
\int_{x_0-\epsilon}^{x_0+\epsilon} \frac{dx}{(x-x_0 \pm \epsilon)(x+x_0 \mp \epsilon)}
\]

since this is an even function of \( x \)

\[
\frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(x-x_0 \pm \epsilon)(x+x_0 \mp \epsilon)}
\]

closing in the upper half plane

\[
2\pi i \frac{1}{2} \cdot \left\{ \frac{-1}{2x_0} \right\} = \pi \text{Im} \cdot \text{Re} \cdot f(x_0)
\]

This integral is \( \pi \text{Im} f(x_0) \)
we see this this integral is purely imaginary - while the principal value contribution is real:

\[ P \int_{0}^{\infty} \frac{dx}{x^2 \cdot \alpha} \]  

This can also be shown by direct computation of the principal value.

**Multivalued functions**

Recall

\[ \ln z \]

\[ z = e^z \]

can be solved to get

\[ \ln z = \ln |z| + i\phi \]

\[ \phi = \arg(z) + 2\pi n \]

If we consider the branch of the ln where \( 0 \leq \phi < 2\pi \) and let \( \phi \) increase by \( 2\pi \) the function does not return to its initial value - instead it returns to its initial value + \( 2\pi i \).

**Definition** A point \( z \) in the complex plane that does not return to its starting value after going around any circle around that point is called a branch point.
Based on this definition, 0 is a branch point of \( \ln z \). Note that
\[
\ln \frac{1}{2} = -\ln z
\]
so \( \infty \) is also a branch point of \( \ln z \).

These are the only two branch points of \( \ln z \).

draw a line in the complex plane connecting the branch points. It is called a branch cut.

Its position is arbitrary.

On this branch, \( 0 \leq \phi < 2\pi \) the function is single valued.
For the \( \ln z \), there are an infinite number of branches that define the function

\[
f_n(z) = \ln |z| + i \left( \theta + 2\pi n \right) \quad 0 \leq \theta < 2\pi
\]

(we could have also used \(-\pi \leq \theta < \pi\) or any other starting point).

\( f_n(z) \)

is analytic in the cut plane. There are many ways to see this. One is to show that the real and imaginary parts satisfy the Cauchy Riemann equations in the cut plane

\[
\begin{align*}
u &= \ln \sqrt{x^2 + y^2} \\
v &= \tan^{-1} \left( \frac{y}{x} \right)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial u}{\partial x} &= \frac{x}{x^2 + y^2} \\
\frac{\partial u}{\partial y} &= \frac{y}{x^2 + y^2} \\
\frac{\partial v}{\partial x} &= \frac{y}{x^2 + y^2} \\
\frac{\partial v}{\partial y} &= \frac{-x}{x^2 + y^2}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial u}{\partial x} &= \frac{1}{\cos u} \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -\frac{1}{\cos u} \frac{\partial v}{\partial x}
\end{align*}
\]

we see that these equations hold on each sheet. For the case of \( \ln z \) there are an infinite number of Riemann sheets.
Next we consider \( \sqrt{z} \)

\[
z = re^{i\phi}
\]

\[
\sqrt{z} = r^{1/2}e^{i\phi/2}
\]

when \( \phi \to 0 \to 2\pi \)

\[
\sqrt{z} \to r^{1/2}e^{i\phi/2}i\pi = -r^{1/2}e^{i\phi/2}
\]

if the phase is increased by another \( 2\pi \)

the function returns to its original value

For \( \sqrt{z} \) \( 0 \) is clearly a branch point. If we consider \( 1/\sqrt{z} \)

it has another branch point at \( z = \infty \), these are the only two branch points of \( \sqrt{z} \).

A branch cut is a line that connects the point.

the function has 2 branches

\[
\sqrt{z}_1 = r^{1/2}e^{i\phi/2} \quad 0 \leq \phi < 2\pi
\]

\[
\sqrt{z}_2 = -r^{1/2}e^{i\phi/2} \quad 0 \leq \phi < 2\pi
\]

each function is analytic in the cut plane.
Similarly

\[ f(z) = z^n = r^n \text{e}^{i\phi/n} \]

This also has branch points at 0 and \( \infty \).

\[ f_1(z) = r^{\frac{n}{n}} \text{e}^{i\phi/n} \quad 0 \leq \phi < 2\pi \]

\[ f_2(z) = e^{\frac{2\pi i}{n}} r^{\frac{n}{n}} \text{e}^{i\phi/n} \quad 0 \leq \phi < 2\pi \]

\[ \vdots \]

\[ f_n(z) = e^{\frac{n-1}{n}(2\pi i)} r^{\frac{n}{n}} \text{e}^{i\phi/n} \quad 0 \leq \phi < 2\pi \]

After the \( n \)th branch, increasing \( \phi \) returns back to the first branch.

Next consider

\[ g(z) = (z^2 - 1)^{1/2} = (z - 1)^{1/2} (z + 1)^{1/2} \]

This function has branch points at \( z = \pm 1 \). Consider

\[ g(z) = f\left(\frac{1}{z}\right) = \left(\frac{1}{z^2} - 1\right)^{1/2} = \left(\frac{1 - z^2}{z}\right)^{1/2} = \left(\frac{(1-z)^2(1+z)}{z^2}\right)^{1/2} \]

This has a simple pole at \( z = 0 \) — which is not a branch point since the value of the function does not change going around \( z = 0 \).
In this case it is possible to choose different cuts

\[ \text{(meet at } \infty) \]

Integrals where integrands have branch cuts.

Cauchy's theorem can still be used with the residue theorem, but the curve cannot cross branch cuts, where the function is not continuous.

Example

\[ I = \int_0^\infty \frac{x^{p-1}}{x^2+1} \, dx \quad \text{for } 0 < p < 2 \]

The integrand of this integral has simple poles at \( x = \pm i \) and branch points at \( 0 \) and \( \infty \). \( (p \neq 1) \)

To perform the integral, it is necessary to choose a branch.
1. Choose the branch cut to be the line from \(0 \to \infty\).

   - Add poles

\[
z^{p-1} = e^{\ln z^{p-1}} = e^{(p-1)\ln z} = e^{(p-1)(\ln r + i(\phi + 2\pi n))} = r^{p-1}e^{i(p-1)(\phi + 2\pi n)}
\]

\[
f(z) = \frac{z^{p-1}}{z^2 + 1} = \frac{r^{p-1}e^{i(p-1)(\phi + 2\pi n)}}{r^2e^{2i\phi} + 1}
\]

To do the integral it is necessary to choose a branch

For \(n = 0\):

\[
f(z) = \frac{r^{p-1}e^{i(p-1)\phi}}{r^2e^{2i\phi} + 1}
\]

Just above the real axis:

\[
f_0(z) = \frac{r^{p-1}}{r^2 + 1}
\]

Just below the real axis:

\[
f_0(z) = \frac{r^{p-1}e^{2\pi i(p-1)}}{r^2 + 1}
\]
To evaluate this integral consider the contour

In this case the curve lies in one branch of the multivalued function

1. In large $|z| \sim |x|^{p-3}$ so the contribution from the large circle is $0$: $r^{p-3} = r^{p-2} \rightarrow 0$
   because $p < 2$ as $r \rightarrow 0$

2. The value of the integral is computed using the residue theorem

$$2\pi i \left[ \frac{e^{i(p-1)\frac{\pi}{2}}}{2i} + \frac{e^{i(p-1)\frac{3\pi}{2}}}{-2i} \right] =$$

$$0 + \int_{0}^{\infty} \frac{x^{p-1}}{x^2 + 1} \, dx + \int_{0}^{\infty} \frac{x^{p-1}}{x^2 + 1} e^{2\pi i (p-1)}$$

$$+ \int_{0}^{2\pi} \frac{e^{i(p-1)\phi}}{e^2 e^{2i\phi} + 1} \, i e^{i\phi} \, d\phi$$

The last integral vanishes as $e \rightarrow 0$
what remains is

\[
\int_0^\infty \frac{x^{p-1}}{x^2+1} \, dx \left( 1 - e^{-2\pi i (p-1)} \right) = \pi e^{i (p-1) \frac{\pi}{2}} \left( 1 - e^{-i (p-1) \pi} \right)
\]

\[
\int_0^\infty \frac{x^{p-1}}{x^2+1} \, dx = \pi \frac{e^{i (p-1) \frac{\pi}{2}} \left( 1 - e^{-i (p-1) \pi} \right)}{e^{i (p-1) \pi} - e^{-i (p-1) \pi}}
\]

\[
= \pi \frac{e^{i (p-1) \frac{\pi}{2}} \left( 1 - e^{-i (p-1) \pi} \right)}{e^{i (p-1) \pi} - e^{-i (p-1) \pi}} = \pi \frac{\sin \left( (p-1) \frac{\pi}{2} \right)}{\sin \left( (p-1) \frac{\pi}{2} \right)}
\]

\[
= \frac{\pi}{2} \frac{\sin \left( (p-1) \frac{\pi}{2} \right)}{\sin \left( (p-1) \frac{\pi}{2} \right)} \cos \left( (p-1) \frac{\pi}{2} \right) = \frac{\pi}{2} \frac{1}{\cos \left( (p-1) \frac{\pi}{2} \right)}
\]

Extending analytic functions

1. Let \( f_1(z) \) and \( f_2(z) \) be analytic in a region \( R \). Let \( S \) be a set in \( R \) with an accumulation point \( z_0 \in R \). Assume \( f_1(z_1) = f_2(z_1) \) for all \( z \in S \).

Then \( f_1(z) = f_2(z) \) in \( R \).

Since \( f_1(z) \) and \( f_2(z) \) are both analytic, \( g(z) = f_1(z) - f_2(z) \) is analytic. By assumption \( g(z) \) is analytic in \( S \),
since \( g(z) \) is analytic it is continuous

\[
g(z) = \lim_{n \to \infty} g(z_n) = 0
\]

This means that \( z_0 \) is a zero of an analytic function that is not isolated.

The only analytic function with an isolated \( 0 \) is the \( 0 \) function\( g(z) = 0 \) on \( R \) or \( f_1(z) = f_2(z) \) on \( R \).

**Corollary**

Since curves in neighborhoods contain accumulation points, analytic functions that agree on curves in neighborhoods are identical.

**Analytic continuation**

Assume \( f(z) \) is analytic in a region \( D \) and is known explicitly at a point \( z_0 \in D \). The function has a convergent power series about \( z_0 \) of radius \( R \).
Consider a curve in $D$ connecting 2 points in $D$. The curve can be covered by a finite number of overlapping circles in $D$. (This is not a trivial result—it is related to the compactness of the curve.)

![Diagram](image)

At each point where the circle intersects the curve, draw a new circle in $D$ and expand about that point. The functions agree on the overlap, so they define a single analytic function. This process can be used to build the function at any point in $D$.

This process is called analytic continuation.