Lecture 21

Matrix representations of vector space

1. \{ \{v_n\}\}_{n=1}^{\infty} \text{ linearly independent if}
\sum_{n=1}^{\infty} c_n \{v_n\} = 0 \implies c_n = 0 \quad n = 1, \ldots, N

2. \{ \{v_n\}\}_{n=1}^{\infty} \text{ span } V \text{ is any vector in } V \text{ can be expressed as}
\{0\} = \sum_{n=1}^{\infty} c_n \{v_n\}

3. \dim V = \max \# \text{ indep vectors}

4. A basis is a linearly independent spanning set

\dim \leq N \text{ indep}

\dim \leq N \text{ span}

\therefore \# \text{ elements in a basis } = \dim \text{ of space}

Representations of vectors \{ \{v_n\}\}_{n=1}^{\infty} \text{ basis}
\{0\} = \sum_{n=1}^{\infty} a_n \{v_n\}

\quad a_n = \text{ components of vector } \{0\} \text{ in basis } \{v_n\}

Note \{0\} is a fixed vector but its components depend on the choice of basis.
we represent this vector by a column of numbers

\[ |a\rangle \rightarrow \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} \]

where the choice of basis is implicit

\[ |c\rangle = (a\rangle + \gamma |b\rangle \rightarrow \left( \begin{array}{c} c_1 \\ c_2 \\ \vdots \\ c_N \end{array} \right) = \left( \begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_N \end{array} \right) + \gamma \left( \begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_N \end{array} \right) \]

\[ \sum (c_n - a_n - \gamma b_n) |v_n\rangle = 0 \]

by independence requires

\[ c_n = a_n + \gamma b_n \]

for each \( n \).

Therefore, \( |a\rangle \) is a vector in basis \( \{|v_n\rangle | n = 1, 2, \ldots, N \} \) with components \( a_n \), \( |b\rangle \)

is another vector with components

\[ |a\rangle = \sum_{n=1}^{N} a_n |v_n\rangle \]

\[ |b\rangle = \sum_{n=1}^{N} c_n |v_n\rangle \]
\[ B | V_n > = \sum_{n} \alpha_n | V_n > b_{n\alpha} \]

\[ \therefore B | a > = B (\sum \alpha_n | V_n >) = \]

\[ \sum B | V_n > a_n = \sum | V_m > b_{mn} a_n \]

It follows that

\[ B | a > = \sum | V_n > c_n = \sum | V_m > b_{nm} a_m \]

\[ \sum | V_n > (c_n - \sum b_{nm} a_m) = 0 \]

so by linear independence

\[ c_n = \sum b_{nm} a_m \]

this can be expressed as a matrix equation

\[
\begin{pmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_n
\end{pmatrix} =
\begin{pmatrix}
  b_{11} & b_{12} & \cdots & b_{1n} \\
  b_{21} & b_{22} & \cdots & b_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{n1} & b_{n2} & \cdots & b_{nn}
\end{pmatrix}
\begin{pmatrix}
  a_1 \\
  a_2 \\
  \vdots \\
  a_n
\end{pmatrix}
\]

( in general \( M \) does not have to equal \( N \). In that case the matrix represents a linear transformation from \( V_M \rightarrow V_N \) (\( NM = \text{dimension} \))
as with the case of vectors, while $B$ represents an operator - the matrix representation depends on the basis.

How do we calculate $b_{nm}$

$$B|v_m\rangle = \sum_n b_{nm} |v_n\rangle$$

calculate

$$\langle v_k | B | v_m \rangle = \sum_n \langle v_k | v_n \rangle b_{nm}$$

define

$$V_{km} = \langle v_k | v_m \rangle$$

$$b_{nm} = \sum V_{mr}^{-1} \langle v_r | B | v_m \rangle$$

note that if $|v_k\rangle$ are independent the matrix $V_{km}$ necessarily has an inverse (we will show this shortly).

If $A, B$ are linear operators with matrix representation $a_{mn}$ $b_{nm}$ then

$$C|v_n\rangle = \sum |v_k\rangle C_{kn} = A(B|v_n\rangle)$$

$$= A \sum |v_k\rangle b_{kn}$$

$$= \sum |v_k\rangle C_{kn} b_{kn}$$
using the independence

\[ C_{nm} = 2 \sum_{k} a_{nk} b_{km} \]

which can be expressed as

\[
\begin{pmatrix}
C_{ii} & C_{in} \\
C_{ni} & C_{nn}
\end{pmatrix}
= \begin{pmatrix}
a_{ii} & a_{in} \\
a_{ni} & a_{nn}
\end{pmatrix}
\begin{pmatrix}
b_{ii} & b_{in} \\
b_{ni} & b_{nn}
\end{pmatrix}
\]

In a number of cases where adjoints are important. It is useful to have matrix representations of adjoints.

Recall

\[ \langle \alpha | A^\dagger | \beta \rangle = \langle A^\dagger \alpha | \beta \rangle \]

\[ \langle \beta | A^\dagger | \alpha \rangle = \langle \alpha | (A^\dagger)^* \beta \rangle \]

\[ | \alpha \rangle = 2 \sum_{n} \langle n | \alpha \rangle \langle n | \alpha \rangle \]

\[ A^\dagger | \alpha \rangle = 2 \sum_{n} \langle n | (A^\dagger)^m \rangle \langle n | \alpha \rangle \]

\[ | \beta \rangle = 2 \sum_{n} \langle n | \beta \rangle \langle n | \beta \rangle \]

\[ \langle \beta | \alpha \rangle = 2 \sum_{n} \langle \beta | < n | \alpha \rangle \langle n | \alpha \rangle \]

\[ \langle \beta | A^\dagger | \alpha \rangle = 2 \sum_{n} \langle \beta | < n | \alpha \rangle \langle n | (A^\dagger)^m \rangle \langle n | \alpha \rangle \]

\[ = \left( \sum_{n} \langle \beta | < n | \alpha \rangle \langle n | A^m \rangle \langle m | \alpha \rangle \right)^* \]

\[ = 2 \sum_{n} \langle \beta | \alpha \rangle \langle \alpha | A^m \rangle \langle n | m \rangle \]
Comparing these expressions for any \( |a> \leftrightarrow |b> \) gives

\[
<V_n | V_m> (A^\dagger)_{mn} = A^x_{mn} <V_n | V_m>
\]

In terms of the matrix \( V \rightarrow V_{nm} = <V_n | V_m> \)

\[
(A^\dagger)_{mn} = (V^{-1})_{m2} (A^x)^T_{2n} V_{nk}
\]

or

\[
A^\dagger = V (A^x)^T V
\]

An orthonormal basis is one where

\[
V_{nm} = <V_n | V_m> = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}
\]

In an orthonormal basis

\[
(A^\dagger)_{mn} = A^x_{nm}
\]

An operator is Hermitian if

\[
A^\dagger = A = \bar{A} \quad A_{mn} = A^\dagger_{nm}
\]

An operator is Unitary if

\[
A^\dagger = A^{-1} = (A^T)^x_{mn} = (A^{-1})_{mn}
\]
Define
\[ \delta_{n_1 n_2} \]
by
\[ \delta_{1 n} = 1 \text{ if completely antisymmetric.} \]

Example:
\[ \delta_{123} = 1 \quad \delta_{123} = -1 \quad \delta_{123} = 1 
\]
\[ \delta_{123} = -1 \quad \delta_{123} = 1 \quad \delta_{123} = -1 \]
all other combinations vanish.

Let\( A_{mn} \) be an \( n \times n \) matrix.

Define
\[ \det(A) = \sum_{n_1 n_2} \delta_{n_1 n_2} A_{n_1 n_2} \]

**Theorem:** If \( \det(A) \neq 0 \) then \( A_{n_1 n_2} \)
are components of \( n \) linearly independent vectors.

By contradiction assume
\[ A_{mn} = \sum_{k+m} c_{k} A_{mn} \]
Assume

\[ \det A = \sum E_{n, m} A_{0n1} A_{m-n, 1} \left( \sum_{k=0}^{\infty} c_k A_{m-n, k} \right) A_{m-n, n} A_{n, n} \]

at least one of the \( c_k = 0 \) for \( k > p \)

\[ A_{0n1} \cdot c_k A_{m-n, k} = A_{e, n} \cdot c_k A_{m-n, k} = A_{e, n} \cdot c_k A_{m-n, k} \]

by relabeling dummy indices, but \( E \) is antisymmetric so interchange \( n \leftrightarrow m \) to this is equal to + end itself

which means \( \det A = 0 \). This gives a contradiction.

A similar argument shows that the columns are components of independent vectors.

This another way to express the determinant.

Let \( \sigma \) be a permutation

\[
\sigma(1) \quad \sigma(n) \\
\eta_1 \quad \eta_n
\]

then one \( N! \) permutations on \( N \) objects.
6. There are \( N \) choices for \( \sigma(1) \)
   There are \( N-1 \) choices for \( \sigma(2) \)
   
   There is 1 choice for \( \sigma(N) \)

5. A permutation is even if it involves an even \# of pairwise exchanges.
   A permutation is odd if it involves an odd \# of pairwise exchanges.

   define \( \text{sgn} \) as:
   \[
   \text{sgn} = \begin{cases} 
   1 & \text{if involves an odd} \\
   0 & \text{if involves an even} \\
   \end{cases}
   \]
   \# of pairwise exchanges

3. Note a permutation can be generated by different \# of exchanges, but they are always odd or even.
   \[
   \begin{pmatrix}
   1 & 2 & 3 \\
   2 & 1 & 3
   \end{pmatrix}
   \begin{pmatrix}
   1 & 2 & 3 \\
   2 & 1 & 3
   \end{pmatrix}
   \]
   \[
   \begin{pmatrix}
   1 & 2 & 3 \\
   3 & 2 & 1
   \end{pmatrix}
   \begin{pmatrix}
   1 & 2 & 3 \\
   3 & 2 & 1
   \end{pmatrix}
   \]
   \[
   \begin{pmatrix}
   3 & 2 & 1 \\
   1 & 2 & 3
   \end{pmatrix}
   \begin{pmatrix}
   3 & 2 & 1 \\
   1 & 2 & 3
   \end{pmatrix}
   \]

   We see both are odd.

   How show \( \sigma \) cannot be both odd and even.
\[ \det(A) = \sum_{\sigma} (-1)^{\sigma} \prod_{i=1}^{n} A_{i\sigma(i)} \]

**Example 3x3**

\[ \det(A) = A_{11} A_{22} A_{33} - A_{11} A_{23} A_{32} + A_{13} A_{21} A_{32} - A_{13} A_{22} A_{31} + A_{12} A_{23} A_{31} - A_{12} A_{21} A_{33} \]

If I treat the \( A_i \) as independent variables, then:

\[ \det(A) = \sum_{i} A_{1i} \frac{\partial}{\partial A_{1i}} \det(A) \]

\[ = A_{11} (A_{22} A_{33} - A_{23} A_{32}) + A_{12} (A_{21} A_{33} - A_{23} A_{31}) + A_{13} (A_{22} A_{31} - A_{21} A_{32}) \]

This clearly generalizes to any dimension:

\[ \det(A) = \sum_{i} A_{1i} \frac{\partial}{\partial A_{1i}} (\det(A)) \quad (r \text{ fixed}) \]
Note $n \neq m$

$$
\sum \frac{\partial \det A}{\partial A_{mr}} = \text{replaces } m^{th} \text{ row by } k^{th} \text{ row}
$$

$$
\sum A_{mr} \frac{\partial \det A}{\partial A_{mr}}
$$

$$
A_{21} \left( \frac{A_{22} A_{33} - A_{23} A_{32}}{\det A} \right) \\
A_{22} \left( A_{21} A_{33} - A_{23} A_{31} \right) \\
A_{23} \left( A_{21} A_{32} - A_{22} A_{31} \right)
$$

we can see in the example all cancel.

$$
\sum \frac{\partial \det A}{\partial A_{mr}} = S_{Rm} \det A
$$

$$
S_{Rm} = \sum A_{Rp} \frac{1}{\det A} \frac{\partial \det A}{\partial A_{mr}}
$$

$$
= \sum \frac{\partial A_{Rm}}{\partial A_{mr}} \left( \ln \det A \right)
$$

$$
A^{-1}_{em} = \frac{1}{\det A} \frac{\partial \det A}{\partial A_{me}}
$$

$$
= \frac{\partial}{\partial A_{me}} \ln(\det A)
$$
The matrix

\[
C_{em} = \frac{\partial \det A}{\partial A_{me}}
\]

is called the cofactor matrix of \( A \)

\[
A^{-1} = \frac{1}{\det A} C
\]

This shows that \( A \) has an inverse if \( \det A \neq 0 \).

Example: \[
\begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix}
\]

\[
\det A = a_{11}a_{22} - a_{12}a_{21}
\]

\[
C_{11} = a_{11}, \quad C_{12} = -a_{12},
\]

\[
C_{21} = a_{11}, \quad C_{22} = -a_{22}
\]

\[
\begin{pmatrix}
  a_{22} - a_{12} \\
  -a_{11}
\end{pmatrix}
\begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix} = \begin{pmatrix}
  \det A \cdot c_{11} \\
  \det A \cdot c_{21}
\end{pmatrix} = \det A \begin{pmatrix}
  c_{11} \\
  c_{21}
\end{pmatrix}
\]

Let \( \det A, \det B \) both be non-zero.

\[
\det AB = \sum_{\sigma(\gamma)}^{2} A_{1m_1}B_{m_1\sigma(\gamma)}A_{\gamma n_\mu}B_{n_\mu\sigma(\nu)}
\]

\[
\sigma, \gamma, m, n
\]
because of the even permutations, the second sum vanishes unless all of the m's are different (indepent vectors)

there are N! combinatorial distinct permutations, to permute

this means we can replace the m sum by a sum over permutations:

\[ \det AB = \sum_{\sigma_1, \sigma_2} A_{\sigma(1)} \sigma(1) \sigma(2) \sigma(3) \cdots \sigma(n) \sigma'(1) \sigma'(2) \cdots \sigma'(n) \]

we introduce \((-1)^{\sigma_1 + \sigma_2} = 1\)

\[ = \sum_{\sigma(1)} A_{\sigma(1)} \lambda_{\sigma(2)} (-1)^{\sigma_1} \]

\[ = \sum_{\sigma_1} (-1)^{\sigma_1 + \sigma_2} \lambda_{\sigma_1} \lambda_{\sigma_2} (-1)^{\sigma_1} \]

\[ = \sum_{\sigma_1} (-1)^{\sigma_1 + \sigma_2} \lambda_{\sigma_1} \lambda_{\sigma_2} \]

\[ = \sum_{\sigma_1} (-1)^{\sigma_1 + \sigma_2} \lambda_{\sigma_1} \lambda_{\sigma_2} \]

here we reorder the \(B\) so \(i\) comes first - that uses the inverse of \(\sigma\) with a phase \((-1)^{\sigma_1}\), \(\sigma'(\sigma \cdot \sigma'(1)) = \sigma'(1)\)

\[ \det AB = \det A \cdot \det B. \]
If \( B \) has an inverse,
\[
\det (BA B^\top) = \det B \det (AB^\top)
\]
\[
\det (AB^\top) \det B = \det (AB^\top B) = \det (A)
\]

If \( A_{mn} = \langle v_m | A | v_n \rangle \) (orthonormal basis)
\[
\tilde{A}_{mn} = \langle w_m | A | w_n \rangle = \langle w_m | v_k \rangle \langle v_k | A | v_n \rangle \langle v_n | w_m \rangle
\]
\[
= \sqrt{\lambda} \sqrt{\lambda}
\]
\[
\det (\tilde{A}) = \det (\sqrt{\lambda} \sqrt{\lambda}) = \det A
\]

We see that the determinant has the same value in any orthonormal basis. It is a property of the operator.

Clearly, \( \det I = 1 \);
\[
l = \det I = \det AA^\top = \det A \cdot \det A^\top
\]
\[
\det A^\top = \frac{1}{\det A}
\]
Change of Basis

Let \( \{ |v_n\rangle \} \) and \( \{ |w_n\rangle \} \) be orthonormal bases, then

\[
|a\rangle = \sum_n |v_n\rangle a_n = \sum_n |w_n\rangle a'_n
\]

\[
\langle v_m | a \rangle = \sum_n d_{mn} a_n = a_m
\]

\[
\langle w_m | a \rangle = \sum_n d_{nm} a'_n = a'_m
\]

\[
|a\rangle = \sum_n |v_n\rangle \langle v_n | a \rangle = \sum_n |w_n\rangle \langle w_n | a \rangle
\]

Since \( |a\rangle \) is arbitrary,

\[
I = \sum_n |v_n\rangle \langle v_n | = \sum_n |w_n\rangle \langle w_n |
\]

\[
I = I^2 = \sum_n |v_n\rangle \langle v_n | w_m\rangle \langle w_m |
\]

\[
|a\rangle = \sum_n |v_n\rangle \langle v_n | w_m\rangle \langle w_m | a \rangle
\]

\[
= \sum_n |v_n\rangle \langle v_n | a \rangle
\]

The coefficients \( \langle v_n | w_m \rangle \) are matrix elements of the operator \( U \)

\[
U = \sum_n |w_n\rangle \langle v_n |
\]

\[
\langle v_n | U | v_k \rangle = \langle v_n | w_k \rangle
\]

\[
\langle w_n | U | w_k \rangle = \langle v_n | w_k \rangle
\[ u = \sum w_n <v_m> \]
\[ <a_1 u_1 b> = <u^t a_1 b> \]
\[ <a_1 u^t_1 b> = <a_1 u_1 b>^* \]
\[ <b_1 u^t_1 b> = \left( \sum <a_1 w_n> <v_m b> \right)^* \]
\[ = \sum <b_1 w_n> <w_m> \]

\[ u^t = \sum w_n <w_m> \]

and

\[ u^t u = \sum w_n <w_m> <w_n> <v_m> \]
\[ = \sum w_n s_{nm} <v_m> \]
\[ = I \]

\[ uu^t = \sum w_n <w_m> <v_n u^t v_m> <w_k> \]
\[ = \sum w_n <w_k> <w_m> \]
\[ = I \]

This shows that any change of orthonormal basis is given by a unitary operator.

Also note if \( \{w_n\} \) is an orthonormal basis \( \{u_n\} = u \{w_n\} \) then \( \{u_n\} \) is an orthonormal basis.

\[ <t_n u^t t_m> = <t_n u w_n> = <w_n u^t t_m>^* \]
\[ = <w_n u^t u^t v_m> = s_{nm} \]
Let $|a\rangle$ be a vector

$|a\rangle = \sum a_n |v_n\rangle \tilde{a}_n$

$\langle b|a\rangle = \langle b|v\rangle \tilde{a}_n = \sum \langle b|v_n\rangle a_n$

$\langle a|b\rangle = \sum \langle a|b|v_n\rangle \tilde{a}_n \tilde{a}_m$

$= \sum \langle v_n|b\rangle \tilde{a}_n \tilde{a}_m$

$\langle a| = \sum \tilde{a}_n \langle v_n|$

If we transform the basis,

$|a\rangle = \sum |v_n\rangle a_n = \sum |w_m\rangle \tilde{a}_m$

$= \sum |v_n\rangle a_n = \sum |v_n\rangle <v_n|w_m\rangle \tilde{a}_m$

$\tilde{a}_n = \sum <v_n|w_m\rangle \tilde{a}_m$

$\tilde{a}_n^* = \sum <v_n|w_m\rangle^* \tilde{a}_m^*$

$= \sum \tilde{a}_m^* <w_m|v_n\rangle$

when the components of a vector transform like

$a_n \rightarrow U_{nm} a_m$

and the dual vector $\tilde{v}$

$\tilde{a}_n^* \rightarrow \tilde{a}_m^* U^*$
\[
\langle a_{1b} \rangle = \sum a_n \langle v_n | v_m \rangle b_m
\]
\[
= \sum a_n b_n
\]
\[
= \sum \tilde{a}_n \langle w_n | w_m \rangle \tilde{b}_m
\]
\[
= \sum \tilde{a}_n \langle w_n | v_k \rangle \langle v_k | w_m \rangle \tilde{b}_m
\]
\[
\tilde{a}_n U_{nk} U_{rm} b_m
\]

In physics, there are objects that transform like products of vectors. Examples are the electromagnetic field strength tensor:

\[
F^{uv} = \begin{pmatrix} 0 & -E_x \\ E_x & 0 & B_y \\ 0 & -B_y & 0 \end{pmatrix}
\]

A 2-particle spin state \( |\mu_1 \mu_2 \rangle \) has many other examples. These quantities are called tensors:

\[
b_n \rightarrow b'_n = U_{nk} b_k
\]

\[
B_{n_1 n_2} \rightarrow B'_{n_1 n_2} = U_{n_1 n_{1'}} U_{n_2 n_{2'}} B_{n_{1'} n_{2'}}
\]

These quantities are called tensors.
The tensor can also have dual indices that transform like linear functionals

\[ B_{\bar{n}_1, \bar{n}_m} \rightarrow B_{\bar{n}_1, \bar{n}_m} = \]

\[ u_{\bar{n}_1}^\dagger \text{ } u_{\bar{n}_m}^\dagger \text{ } U_{n_1 \bar{n}_1} \text{ } \cdots \text{ } U_{n_m \bar{n}_m} \text{ } B_{k_1, k_m} \]

Summing over an index and a dual index:

\[ \sum \text{ } B_{n_1, n_2, \bar{n}_m} = \]

\[ \sum \text{ } u_{k_1}^\dagger \text{ } U_{n_1 \bar{n}_1} \text{ } u_{k_2}^\dagger \text{ } U_{n_2 \bar{n}_2} \text{ } \cdots \text{ } u_{k_m}^\dagger \text{ } U_{n_m \bar{n}_m} \text{ } B_{k_1 k_2, k_m} \]

We see that the sum is preserved and the tensor has lost an index and one dual index.

Some essential properties of tensors:

A tensor that has \( n \) indices and \( m \) dual indices is called a rank \( n:m \) tensor.
summing over an index and a dual index is called a contraction. It transforms a rank \( n \) \( m \) tensor to a rank \( n-1 \) \( m-1 \) tensor.

In general, the tensors transform with respect to different groups of transformation.

In special relativity, the \( U \) are replaced by Lorentz transformations.

In many particle systems, the \( U \) is replaced by the rotation matrices.

Eigenvalue problems. The eigenvalue problem

\[ A \lambda_\alpha = \lambda_\alpha \]

can be solved using matrix representations. Let \( |\lambda_n\rangle \) be an orthonormal basis,

\[ \sum_n \langle \lambda_n | A | \lambda_n \rangle \langle \lambda_n | \lambda \rangle = \lambda \langle \lambda_n | \lambda \rangle \]

\[
\begin{pmatrix}
A_{\alpha\beta} & A_{\alpha\nu} \\
A_{\beta\alpha} & A_{\beta\nu}
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_\nu
\end{pmatrix} = \lambda
\begin{pmatrix}
\alpha_1 \\
\alpha_\nu
\end{pmatrix}
\]
we write this as

\[
\begin{pmatrix}
\lambda - A_{11} & -A_{12} & -A_{13} \\
-A_{21} & \lambda - A_{22} & -A_{23} \\
-A_{31} & -A_{32} & \lambda - A_{33}
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix} = 0
\]

If this matrix has an inverse, applying the inverse gives

\[
\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

This is a trivial solution of the equation. For a solution to require that the matrix has no inverse - this means

\[
\det(\lambda I - A) = 0
\]

It is easy to see from the definition of \(\det\) that

\[
\det(\lambda I - A) = \lambda^N + \sum_{k=0}^{N-1} C_k \lambda^k
\]

This is a degree \(N\) polynomial in \(\lambda\).
by the fundamental theorem of linear algebra this polynomial has \( N \) roots. These are the eigenvalues.

Cayley-Hamilton Theorem

\[
P(\lambda) = \det(\lambda I - A)
\]

\[
P(A) = 0
\]

To prove this,

\[
P(\lambda) s_{ii} = \sum \frac{\partial \det(\lambda I - A)}{\partial m_{ij}} (\lambda I - A)_{ii}
\]

Setting \( \lambda = A \)

\[
= \sum \frac{\partial \det(\lambda I - A)}{\partial m_{ii}} (A - A)_{ii} = 0
\]

\[
P(A) = \prod (\lambda - \lambda_i)
\]

\[
0 = \prod (A - \lambda_i) = A^N + \sum_{n=0}^{N-1} c_n A^n
\]

\[
A^N = -\sum_{n=0}^{N-1} c_n A^n
\]

By induction assume

\[
A^M = \sum_{k=0}^{M-1} c_k A^k
\]

\[
A^{M+1} = \sum_{k=0}^{M} c_k A^{k+1} = \sum_{k=0}^{M} d_k A^{k+1} + d_{M+1} \left( -\sum_{n=0}^{N-1} c_n A^n \right)
\]
$$A^{N+1} = -d_{N-1} C_0 + \sum_{k=1}^{K-1} \left( d_{k-1} - d_{k+1} C_0 \right) A^k$$

This means that any operator that can be represented as a Cauchy sequence of polynomials can be expressed as a degree $N-1$ polynomial in $A$. 