29:171 - Hour Exam #1

1. Let \( z = x + iy \) and \( f(z) = x^3 + i3x^2y - 3y^2x - iy^3 \)
   
   a. Show that this function is analytic.
   
   b. Show that the real and imaginary parts of this function satisfy Laplace’s equation \( (\nabla^2 g(x, y) = 0) \).
   
   c. Is \( h(z) = x^3 + i6x^2y - 12y^2x - iy^3 \) analytic?

2. Calculate the complex line integral

\[
\int_c (z^3 - 4z^2 + 1)dz
\]

where \( c \) is the path composed of straight line segments from \( p_1 \) to \( p_2 \), \( p_2 \) to \( p_3 \), \( p_3 \) to \( p_4 \), where \( p_1 = 1, p_2 = 1 + 2i, p_3 = -2 + 2i, \) and \( p_4 = -2 \).

How does this integral compare to the integral of the same function along the real axis from -2 to 1?

3. Assume that \( f(z) \) is an entire function with the property that \( |f(z)| < C|z|^2 \). Prove that \( f(z) \) must be a polynomial of degree less than or equal to two.

4. Consider the integral

\[
\int_0^\infty \frac{x^2}{(x^2 + 4)(x^2 + 9)}dx
\]

a. Find the poles of the integrand if \( x \) is replaced by a complex value \( z \).

b. Compute this integral.