1. Let $H$ be a Hermitian operator. Let $|m\rangle$ and $|n\rangle$ be eigenvectors of $H$ with eigenvalues $\lambda_m$ and $\lambda_n$.

   a. Show that $\lambda_m$ is real.
   b. Show that $\langle m|n \rangle = 0$ if $\lambda_m \neq \lambda_n$.
   c. Assume that $\lambda_m = \lambda_n$ and $\langle m|n \rangle = \alpha \neq 0$. Find a set of orthonormal eigenvectors with the same eigenvalue.

2. Let $A$ and $B$ be positive operators. Assume that $[A, B] = 0$.

   a. Prove that $AB$ is positive.
   b. Show that $e^{-A}$ is positive.

3. Let $|a\rangle$ and $|b\rangle$ be vectors in an inner product space. Let $P$ be an orthogonal projection operator ($P^2 = P$, $P^\dagger = P$).

   a. Prove that $P$ is a positive operator.
   b. Prove
      \[ |\langle a|P|b \rangle|^2 \leq |\langle a|P|a \rangle||\langle b|P|b \rangle| \]
   c. Is $I - P$ positive?

4. Let $\{|n\rangle\}_{n=1}^{N}$ and $\{|\bar{n}\rangle\}_{n=1}^{N}$ be orthonormal bases in $N$-dimensional inner product space.

   a. Let $W := \sum_{n=1}^{N} |n\rangle\langle \bar{n}|$. Calculate $W^\dagger$
   b. Show that $W$ is unitary.
   c. Assume that $N > 5$. Show that $P = \sum_{n=1}^{4} |n\rangle\langle n|$ is an orthogonal projection operator.
   d. Show that $M = \sum_{n=1}^{N-1} |n+1\rangle\langle n|$ is a nilpotent operator operator.