1. The Green’s function for a harmonic oscillator Hamiltonian has the form

\[ G(x, y) = \sum_{n=0}^{\infty} \frac{\phi_n(x)\phi_n^*(y)}{n + \frac{1}{2}} \]

where the \( \{\phi_n(x)\} \) is a complete set of orthonormal functions on \([-\infty, \infty]\\):

\[ \int_{-\infty}^{\infty} \phi_n^*(x)\phi_m(x)dx = \delta_{mn}. \]

a. Is \( G(x, y) \) compact?

b. Is \( G(x, y) \) a Hilbert Schmidt operator?

c. The operator \( L_x \) satisfies

\[ L_x G(x, y) = \delta(x - y). \]

What are the eigenvalues of \( L_x \) and eigenfunctions of \( L_x \).

d. Are there any solutions of the homogeneous equation

\[ L_x \psi(x) = 0 \]

satisfying the boundary conditions at \( x = \pm \infty \).

2. Let

\[ L_x u(x) = \frac{d^2 u(x)}{dx^2} + \eta^2 u(x) \]

on the interval \([a, b]\\).

a. Find independent solutions to the homogeneous equation \( L_x u(x) = 0 \)

b. Calculate the Wronskian of these solutions, verify that it does not vanish.

c. Find the Green’s function for \( L_x \) satisfying \( u(a) = u(b) = 0 \)

d. Solve

\[ L_x u(x) = e^x \]

with the boundary conditions \( u(a) = u(b) = 0 \).

3. Let

\[ L_x = \frac{d^2}{dx^2} + \frac{cx - 2}{x} \frac{d}{dx} + \frac{2}{x^2} \]

a. Find the indicial equation for this operator.

b. Find the roots of the indicial equation.
c. For the larger of the two roots find the equation relating the \( n + 1 \)-st term in the series solution to the \( n \)-th term.

3. An integral representation for the Hypergeometric function is

\[
F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_1^\infty dt (z - t)^{-a} t^{a-c} (t - 1)^{c-b-1}
\]

a. Make the variable change \( t' = 1/t \) and construct another integral representation.

b. For what values of \( a, b, c \) is the representation in part a. defined?

c. For the case that \( a \) is an integer \( n \) use the integral representation above to find the power series representation of \( F(n, b, c; z) \)

Useful formulas

\[
\sin(a) \cos(b) \pm \sin(b) \cos(a) = \sin(a \pm b)
\]

\[
\cos(a) \cos(b) \pm \sin(b) \sin(a) = \cos(a \mp b)
\]